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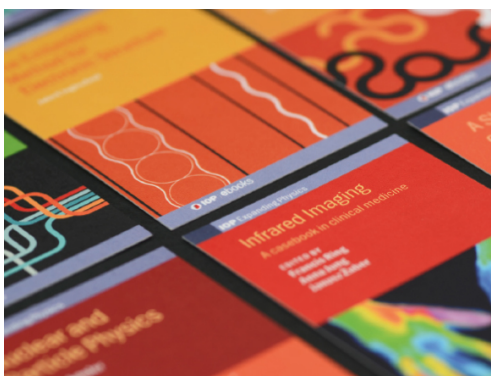
Coordinately assisted distillation of quantum coherence in multipartite system

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Coordinately assisted distillation of quantum coherence in multipartite system

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E-mail: sunzhe@hznu.edu.cn**Keywords:** quantum information, quantum resource theory, quantum optics

Abstract

We investigate the issue of assisted coherence distillation in the asymptotic limit, by coordinately performing the identical local operations on the auxiliary systems of each copy. When we further restrict the coordinate operations to projective measurements, the distillation process branches into many sub-processes. Finally, a computable measure of the assisted distillable coherence is derived as the maximal average coherence of the residual states with the maximization taken over all the projective measurements on the auxiliary. The measure can be conveniently used to evaluate the assisted distillable coherence in experiments, especially suitable for the case that the system and its auxiliary are in mixed states. By using the measure, we for the first time study the assisted coherence distillation in multipartite systems. Monogamy-like inequalities are derived to constrain the distribution of the assisted distillable coherence in the subsystems. Taking a three-qubit system for example, we experimentally prepare two types of tripartite correlated states, i.e., the W -type and GHZ-type states in a linear optical setup, and experimentally test the assisted distillable coherence. Theoretical and experimental results agree well to verify the distribution inequalities given by us. Three measures of multipartite quantum correlation are also considered. The close relationship between the assisted coherence distillation and the multipartite correlation is revealed.

1. Introduction

Quantum coherence, as the fundamental feature of quantum mechanics and a kind of resource [1, 2], is widely used in quantum information processing [3], quantum computation, quantum algorithm [4, 5], quantum metrology [6–9], and quantum thermodynamics [10, 11]. It is the main reason why the quantum world is different from the classical world [12].

In order to quantify coherence [13], one needs a set of reference bases $\{|i\rangle, i = 0, 1, 2, \dots\}$, based on which, the class of incoherent states \mathcal{I} is defined with diagonal density matrices, i.e., $\sum_i \rho_i |i\rangle\langle i| \in \mathcal{I}$. Following this, incoherent operations (IOs) act unchangeably on the assemblage of all incoherent states and satisfy the map $\Lambda_{\text{IO}}(\mathcal{I}) \subseteq \mathcal{I}$. Different types of IOs are proposed in [13–17]. A common measure of coherence for a state ρ is defined by the relative entropy [13], $C_r(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho||\sigma)$, to characterize the minimal distance of ρ to the class of incoherent states \mathcal{I} . One of the most operational measure of the coherence is the distillable coherence which is similar to the framework of the distillable entanglement [18, 19], and was introduced in [20] at the asymptotic limit by considering infinite copies of the state. The optimal rate of a state ρ in a coherence distillation process, defined as the distillable coherence $C_d(\rho)$, is evaluated analytically $C_d(\rho) = C_r(\rho)$ [20, 21]. However, in experiments, it is a huge challenge to collectively manipulate a large number of state copies, and to achieve the asymptotic limit. Therefore, a kind of one-shot coherence distillation was proposed [22] which also provided the possibility of the experimental study. This one-shot scenario was experimentally demonstrated based on a linear optical system [23], where a kind of N -dimensional ($N \geq 2$) IOs were realized.

On the other hand, the asymptotic scenario of coherence distillation was developed into bipartite system ρ_{AB} , where only the operations performed on the second party (B) are restricted to IOs, and the classical communication is allowed between the two parties. These sets of operations are called local quantum-incoherent operations and classical communications (LQICC). Following the LQICC, the concept of the assisted coherence distillation was established in asymptotic settings [24]. The assisted distillation rate R of subsystem ρ_B is bounded by quantum-incoherence (QI) relative entropy $C_r^{A|B}(\rho_{AB})$. For pure states $|\Psi\rangle_{AB}$, the upper bound is accessible, while for mixed states ρ_{AB} , it is still an open question whether the upper bound can be achieved. The experimental study of the assisted coherence distillation was reported in [25], where the authors employed a one-copy scenario. To overcome the difficulty of the experimental demonstration, the nonasymptotic settings of the assisted coherence distillation were proposed [26, 27]. Different from the distillation framework above, in [28] the authors introduced a scenario of steering-induced coherence, which is defined on the eigenvectors of the considered system, and has been conveniently used in open systems [29].

Quantum coherence in multipartite and multilevel systems has been attracted much attention in the last decade. A necessary and sufficient analytical criterion was reported in [30] to verify the presence of multilevel coherence. The problems of the quantum coherence distribution among the constituent subsystems were considered in [31, 32]. The conversion between quantum coherence and quantum correlation was studied in [33, 34]. We find that an issue worthy of study is what distribution law of the assisted distillable coherence obeys in the multipartite system when a subsystem is chosen as the assistance. Furthermore, the relation between the multipartite correlation and the assisted coherence distillation is also worth investigating.

In this work, we take into account a class of operations named as coordinate local quantum-incoherent operations and classical communication (CoLQICC), which is a subset of LQICC. Based on the CoLQICC operations, we introduce the coordinate asymptotic scenario of the assisted coherence distillation and derive a computable measure to evaluate the assisted distillable coherence. By using the measure, we for the first time investigate the distribution of the assisted distillable coherence in multipartite systems. We develop a monogamy-like inequality to reveal that the assisted distillable coherence of each subsystem should be constrained by that of the remaining combined system. Taking a three-qubit system for example, we experimentally measure the assisted distillable coherence and verify the inequality relation. By numerical calculations, we show the relationship between the multipartite correlation and the assisted coherence distillation.

2. Coordinately assisted distillation of coherence

For a bipartite system of Alice and Bob sharing the state ρ_{AB} , the aim of assisted coherence distillation is to concentrate the coherence resource on Bob's side by allowing Alice to perform arbitrary quantum operations. To quantify the optimal rate of assisted distillable coherence, the definition is given as follows [24]:

$$C_d^{A|B}(\rho) = \sup \left\{ \mathcal{R} : \liminf_{n \rightarrow \infty} \inf_{\Lambda} \|\Lambda(\rho^{\otimes n}) - \Phi_2^{\otimes \lfloor n\mathcal{R} \rfloor}\|_1 = 0 \right\}, \quad (1)$$

where the infimum is taken over all LQICC operations Λ and $\lfloor x \rfloor$ returns the maximum integer no larger than x . $\|O\|_1 = \text{Tr} \sqrt{O^\dagger O}$ is the trace norm. In D -dimensional Hilbert space \mathcal{H} , the maximal coherent resource state is $|\Phi_D\rangle \equiv \sum_{i=0}^{D-1} |i\rangle / \sqrt{D}$, and $\Phi_2 \equiv |\Phi_2\rangle\langle\Phi_2|$ denotes the density matrix of the two-dimensional maximal coherent state. It has been proved that, there is an upper bound of the assisted distillable coherence, i.e., $C_d^{A|B}(\rho_{AB}) \leq C_r^{A|B}(\rho_{AB})$, with the definition $C_r^{A|B}(\rho_{AB}) = S(\Delta^B \rho_{AB}) - S(\rho_{AB})$, and $\Delta^B(\rho_{AB}) \equiv \sum_i (\mathbb{I}_A \otimes |i\rangle_B \langle i|) \rho (\mathbb{I}_A \otimes |i\rangle_B \langle i|)$ with \mathbb{I} being a identity matrix [24]. For a pure bipartite state, the equal sign holds, however, for a mixed state, whether the upper bound can be achieved is still unknown.

In the conventional asymptotic scenario of the coherence distillation, many independent and identically distributed (i.i.d.) copies of the initial resource are to be converted into many i.i.d. copies of the target state. Collective operations are needed to be performed on all the state copies to reach the optimal interconversion rate. Therefore, it is a huge challenge for the current technology to experimentally demonstrate the coherence distillation process. Even for theoretical research, it is still difficult to prove the reachability of the upper bound for a mixed state in the asymptotic scenario.

To derive an effective measure to evaluate the assisted distillable coherence in the asymptotic framework, we introduce a new scenario by considering a type of operation named as CoLQICC. The operation of CoLQICC proposed by us consists of two parts: (i) identical local measurements (operations) on Alice's side are coordinately and separably performed on each copy of the resource state. Let the mapping $\Lambda_{A:\text{CoQ}}$ denote the operation on Alice, for many copies of the state, there should be $\Lambda_{A:\text{CoQ}}^{\otimes n} \rho_{AB}^{\otimes n} = (\Lambda_{A:\text{CoQ}} \rho_{AB})^{\otimes n}$.

A similar setting can be found in [35]; (ii) incoherence operations, denoted by the mapping Λ_B^{IO} , working on the Bob's side. In our consideration, the operations of Λ_B^{IO} will collectively act on the copies of Bob's residual states. Therefore, the CoLQICC can be described by a complete mapping, i.e., $\Lambda_{\text{CoQ}} \equiv \Lambda_B^{\text{IO}} \circ \Lambda_{A:\text{CoQ}}^{\otimes n}$.

Under the CoLQICC, we define the optimal rate of the coordinately assisted coherence distillation as:

$$C_{\text{CoQ}}^{\text{A|B}}(\rho) = \sup \left\{ \mathcal{R} : \liminf_{n \rightarrow \infty} \|\Lambda_{\text{CoQ}}(\rho^{\otimes n}) - \Phi_2^{\otimes \lfloor n\mathcal{R} \rfloor}\|_1 = 0 \right\}, \quad (2)$$

where the infimum is taken over all the CoLQICC operators Λ_{CoQ} . Obviously, when the state of Alice and Bob is in a product form, i.e., $\rho_{AB} = \rho_A \otimes \rho_B$, the assist system A has no effects, then the assisted distillable coherence $C_d^{\text{A|B}}(\rho)$ transforms to the distillable coherence $C_d(\rho_B)$. In order to obtain a computable measure, we further simplify the operations on Alice to orthogonal projective measurements $\{\Xi_A^i\}$, the operators satisfy $\text{Tr}(\Xi_A^i \Xi_A^k) = \delta_{ik}$, $\sum_i \Xi_A^i = \mathbb{I}_A$, and $(\Xi_A^i)^2 = \Xi_A^i$. In this setting, $\Lambda_{A:\text{CoQ}}^{\otimes n} \rho_{AB}^{\otimes n} = \sum_i (\Xi_A^i \otimes \mathbb{I}_B)^{\otimes n} \rho_{AB}^{\otimes n} (\Xi_A^i \otimes \mathbb{I}_B)^{\otimes n}$. Thus, in the following sections, we emphasize the projective measurements by restricting the CoLQICC operations to the coordinate local *projective-incoherent* operations and classical communication (CoLPICC).

Lemma 1. *The optimal rate of the assisted coherence distillation under the proposed CoLPICC operations, can be expressed as follows:*

$$C_{\text{CoP}}^{\text{A|B}}(\rho_{AB}) = \max_{\{\Xi_A^i\}} \sum_i P_i \sup \{ R_i : \liminf_{n \rightarrow \infty} \|\Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - \Phi_2^{\otimes \lfloor nR_i \rfloor}\|_1 = 0 \}, \quad (3)$$

where P_i is the probability distribution and ρ_B^i is the residual density with the definitions:

$$\begin{aligned} P_i &= \text{Tr}(\Xi_A^i \otimes \mathbb{I}_B \rho_{AB}), \\ \rho_B^i &= \frac{\text{Tr}_A(\Xi_A^i \otimes \mathbb{I}_B \rho_{AB})}{P_i}. \end{aligned} \quad (4)$$

The maximum is taken over all the Alice's projective measurements and the infimum is taken over the IOs on Bob's side. The rate of the assisted coherence distillation is a probabilistic sum of all the subprocesses. Finally, the rate of coordinately assisted coherence distillation becomes $\mathcal{R} = \max_{\Xi_A} \sum_i P_i R_i$ (proof details are shown in appendix A).

Our results reveal that different sets of operations performed on Alice will provide different final distillation rates. We design a new scenario of coordinately performing identical projective measurements on each copy of Alice, which will help us to derive a computable measure of the assisted distillable coherence available in the asymptotic limit. The measurements on Alice's side cause the coherence distillation process to branch into several sub-processes, each of which corresponds to a distillation rate R_i . Since $\text{CoLPICC} \subset \text{CoLQICC} \subset \text{LQICC}$, one has the relation $C_{\text{CoP}}^{\text{A|B}}(\rho_{AB}) \leq C_{\text{CoQ}}^{\text{A|B}}(\rho_{AB}) \leq C_d^{\text{A|B}}(\rho_{AB}) \leq C_r^{\text{A|B}}(\rho_{AB})$. $C_{\text{CoP}}^{\text{A|B}}$ and $C_{\text{CoQ}}^{\text{A|B}}$ correspond to the different sets CoLPICC and CoLQICC, respectively.

Theorem 1. *With the proposed CoLPICC operators, the optimal rate of the coordinately assisted coherence distillation has an explicit solution:*

$$C_{\text{CoP}}^{\text{A|B}}(\rho_{AB}) = \max_{\{\Xi_A^i\}} \sum_i P_i C_r(\rho_B^i), \quad (5)$$

with the definitions of P_i and ρ_B^i in equation (4) and the maximum is taken over all the projective measurements $\{\Xi_A^i\}$ on Alice. Perhaps it seems quite restrictive that the local operations are set to projective measurements. The reasons are that: (i) the identical local projective measurements satisfy our requirement well to derive a reachable and computable solution of the assisted coherence distillation in the asymptotic limit, (ii) projective measurements are particularly easy to operate and are widely used in theoretical and experimental studies. It makes the choice more reasonable and easy to accept. Certainly, other types of measurements can be considered, however, it will reduce the possibility of obtaining the computable formula.

Importantly, the measure in equation (5) is suitable for the case that ρ_{AB} is a mixed state, and which is convenient to be tested in experiments. As is known that for mixed states, there is still a lack of the measure, derived in the asymptotic framework, to evaluate the assisted distillable coherence. Whether the analytical upper bound $C_r^{\text{A|B}}(\rho_{AB})$ can be achieved for mixed states is still unknown. Now, we show that $C_{\text{CoP}}^{\text{A|B}}(\rho_{AB})$ can act as an affirmative and reachable measure for mixed states. In appendix B, we prove in detail that for a mixed state, Bob can asymptotically obtain the distillable coherence with the rate of $C_{\text{CoP}}^{\text{A|B}}(\rho_{AB})$ when proper

projective measurements performed on Alice’s side. Our results provide an operational interpretation of the average relative entropy of coherence, which should not be simply understood as a one-copy scenario to approximately evaluate the assisted distillable coherence [25], but a determined reachable measure derived in the asymptotic limit. The proof details of theorem 1 can be found in appendix B, where we first prove the upper bound of the distillation rate is the average of the relative entropy of coherence. Then we prove that the upper bound can be reached by using the typical sequence technique.

Certainly, the CoLQICC operation introduced by us still includes a collective operation on the copies of Bob, and thus this scenario does not overcome the difficulties in the experimental demonstration of the asymptotic distillation process. However, the measure in equation (5), derived through the scenario of CoLQICC, is suitable for the case that one focuses on evaluating the amount of the assisted distillable coherence available in the asymptotic limit. He can prepare one-copy resource state and implement optimal projective measurements on Alice’s side and perform tomography on Bob’s side to evaluate the final distillable coherence.

For a pure state density $\Psi_{AB} \equiv |\Psi_{AB}\rangle\langle\Psi_{AB}|$, through the local measurements on Alice together with the communications with Bob, any possible pure decomposition of ρ_B can be obtained, i.e., $\rho_B = \sum_i p_i \Psi_B^i$ for any set of $\{p_i\}$ and the corresponding pure state density is $\Psi_B^i \equiv |\Psi_B^i\rangle\langle\Psi_B^i|$. Therefore, based on the definition in equation (5), we have $C_{\text{CoP}}^{A|B}(\Psi_{AB}) = \max_{\{\Xi_A\}} \sum_i P_i C_r(\Psi_B^i) = \max_{\{\Xi_A\}} \sum_i P_i S(\Delta\Psi_B^i)$, which is identical to the concept of coherence of assistance (COA) $C_a(\rho_B)$ [24]. Moreover, one can find $C_{\text{CoP}}^{A|B}(\Psi_{AB}) \leq C_d^{A|B}(\Psi_{AB}) = C_r^{A|B}(\Psi_{AB}) = S(\Delta^B\Psi_{AB}) = S(\Delta\rho_B)$. Now let us discuss two special cases of pure states:

- (a) The dimension of subsystem B is $\dim(\mathcal{H}_B) = 2$, then one has $C_a(\rho_B) = S(\Delta\rho_B)$ [24]. Consequently, we have

$$C_{\text{CoP}}^{A|B}(\Psi_{AB}) = C_r^{A|B}(\Psi_{AB}) = S(\Delta\rho_B). \tag{6}$$

- (b) The dimension of auxiliary system (Alice) is $\dim(\mathcal{H}_A) = 2$ and that of Bob is $\dim(\mathcal{H}_B) = n$ ($n > 2$). For a set of reference basis $\{|i\rangle\}$, where the quantum coherence is defined. If the Schmidt decomposition of $|\Psi_{AB}\rangle$ can be written as follows:

$$|\Psi_{AB}\rangle = \sqrt{\lambda_1}|\phi_A^1\rangle \left(\sum_{j \neq i} |j\rangle_B \right) + \sqrt{\lambda_2}|\phi_A^2\rangle |i\rangle_B, \tag{7}$$

where $\langle\phi_A^2|\phi_A^1\rangle = 0$. Then by performing the projective measurement of $\{(|\phi_A^1\rangle \pm |\phi_A^2\rangle)/\sqrt{2}\}$ on Alice, one can easily obtain $C_{\text{CoP}}^{A|B}(\Psi_{AB}) = S(\Delta\rho_B)$. The expression in equation (7) also gives an answer to the remaining issue in [24] that for which kind of high-dimensional pure states, the assisted of coherence (COA), i.e., $C_a(\rho_B) = C_{\text{CoP}}^{A|B}$ for pure states in this work, is equal to the regularized COA for the infinite copies of the state.

Let us expand to the cases of multipartite systems, and take a tripartite pure state for example. If the Schmidt decomposition of a pure state $|\Psi\rangle_{ABC}$ with the condition $\dim(\mathcal{H}_A) = 2$, can be presented as:

$$|\Psi_{ABC}\rangle = \sqrt{\lambda_1}|\phi_A^1\rangle \left(\sum_{\langle mn|ij\rangle=0} |mn\rangle_{BC} \right) + \sqrt{\lambda_2}|\phi_A^2\rangle |ij\rangle_{BC}, \tag{8}$$

where $\{|ij\rangle\}$ denotes a set of reference basis and $\langle\phi_A^2|\phi_A^1\rangle = 0$, we also have the similar equality in equation (6) that:

$$C_{\text{CoP}}^{A|BC}(|\Psi_{ABC}\rangle) = S(\Delta\rho_{BC}). \tag{9}$$

For example, the GHZ-type and W -type states satisfy the decomposition in equation (8), thus the above equality holds.

Assisted coherence distillation in multipartite systems. In the following sections, we will discuss the problems of coordinately assisted coherence distillation in multipartite systems. Let us start from the tripartite case.

Theorem 2. *In tripartite system, for a pure state $|\Psi_{ABC}\rangle$ satisfying the condition in equation (8) and with the dimension of the auxiliary system $\dim(\mathcal{H}_A) = 2$, the following inequality holds,*

$$C_{\text{CoP}}^{A|BC}(|\Psi_{ABC}\rangle) \geq C_{\text{CoP}}^{A|B}(|\Psi_{ABC}\rangle) + C_{\text{CoP}}^{A|C}(|\Psi_{ABC}\rangle), \tag{10}$$

where the first process $C_{CoP}^{A|BC}(\Psi_{ABC}) = \max_{\{\Xi_A^i\}} \sum_i P_{BC}^i C_r(\rho_{BC}^i)$ with $\rho_{BC}^i = \text{Tr}_A(\Xi_A^i \otimes \mathbb{I}_{BC} \rho_{ABC}) / P_{BC}^i$ and $P_{BC}^i = \text{Tr}(\Xi_A^i \otimes \mathbb{I}_{BC} \rho_{ABC})$. The second process $C_{CoP}^{A|B}(\Psi_{ABC}) = \max_{\{\Gamma_A^j\}} \sum_j P_B^j C_r(\rho_B^j)$ with $\rho_B^j = \text{Tr}_{AC}(\Gamma_A^j \otimes \mathbb{I}_{BC} \rho_{ABC}) / P_B^j$ and $P_B^j = \text{Tr}(\Gamma_A^j \otimes \mathbb{I}_{BC} \rho_{ABC})$. The third process $C_{CoP}^{A|C}(\Psi_{ABC}) = \max_{\{\Theta_A^k\}} \sum_k P_C^k C_r(\rho_C^k)$ with $\rho_C^k = \text{Tr}_{AB}(\Theta_A^k \otimes \mathbb{I}_{BC} \rho_{ABC}) / P_C^k$ and $P_C^k = \text{Tr}(\Theta_A^k \otimes \mathbb{I}_{BC} \rho_{ABC})$. Obviously, when the state is in a product form, i.e., $|\Psi_{ABC}\rangle = |\Psi_{AB}\rangle \otimes |\Psi_C\rangle$, $|\Psi_{ABC}\rangle = |\Psi_{AC}\rangle \otimes |\Psi_B\rangle$, or $|\Psi_{ABC}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle \otimes |\Psi_C\rangle$ the equality holds.

Note that there are actually three optimization processes in the inequality of equation (10), which are realized by choosing proper projective measurements Ξ_A^i , Γ_A^j , and Θ_A^k to achieve the maximal values of $C_{CoP}^{A|BC}$, $C_{CoP}^{A|B}$ and $C_{CoP}^{A|C}$, respectively. The proof of theorem 2 is shown in appendix C. The theorem reveals a distribution formula of the coordinately assisted coherence distillation in a tripartite system. The monogamy-like inequality in equation (10) implies that the process of distilling coherence on the combined subsystem BC with assistant A cannot always be divided into two independent subprocesses, i.e., distilling coherence in each subsystem B and C with assistant A. In other words, it points out that the assisted distillable coherence available to each subsystem B (or C) should be constrained by that available to the remaining combined system BC.

When considering the multipartite case of $N > 3$, for a pure state satisfying the condition by extending equation (8) to the multipartite cases, also with $\dim(\mathcal{H}_A) = 2$, then the following inequality holds:

$$C_{CoP}^{A|B_1 B_2 \dots B_N}(\rho_{AB_1 B_2 \dots B_N}) \geq \sum_{\alpha=1}^N C_{CoP}^{A|B_\alpha}(\rho_{AB_1 B_2 \dots B_N}), \tag{11}$$

where each $C_{CoP}^{A|B_\alpha}$ is obtained by performing the corresponding optimal measurement $\Xi_{A,opt}^\alpha$ on system A. The inequality reveals that the assisted distillable coherence of each subsystems should be constrained by that of the remaining combined part.

Theorem 3. For a general state $\rho_{AB_1 B_2 \dots B_N}$ (either pure or mixed), the following inequality holds:

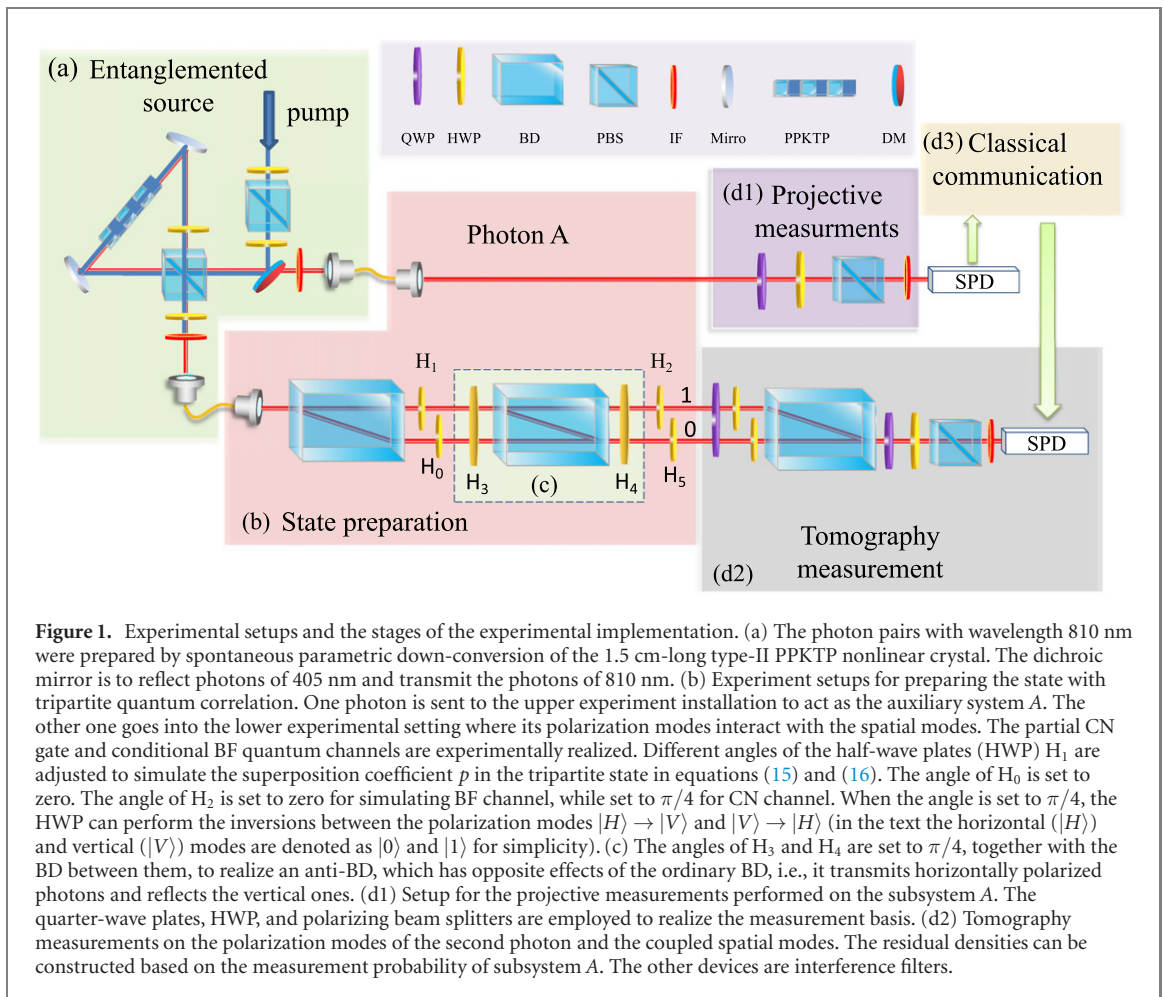
$$C_{CoP}^{A|B_1 \dots B_N}(\rho_{AB_1 \dots B_N}) \geq \max_{\{\Xi_A^i\}} \sum_i P_i \left(\sum_{\alpha=1}^N C_r(\rho_{B_\alpha}^i) \right). \tag{12}$$

Note that the inequality above describes a different assist process from that in theorem 2. Here, Alice only performs the optimal measurement $\Xi_{A,opt}^i$ once to achieve the maximal average of the sum of the distillable coherence of the residual states corresponding to each subsystem B_α . Obviously, when the state is in a product form, e.g., $\rho_{AB_1 \dots B_N} = \rho_{AB_1} \otimes \rho_{B_2} \dots \otimes \rho_{B_N}$ (i.e., at most a pair of subsystems are related) the equality holds. The detailed proof can be found in appendix D. With the help of the measure in equation (5) and the monogamy-like inequalities in theorems 2 and 3, we can experimentally test the distribution relationship of the assisted distillable coherence based on a linear optical setup.

It is known that, the assisted coherence distillation is closely related to the quantum correlation. For a bipartite system AB, it is easy to find that if they share a product state, the coherence resource of Bob will not be changed, no matter what operations are performed on Alice, i.e., the assist system A provides no effect. However, if there exists a proper quantum correlation between the subsystems, the operations on Alice will induce an increase of the distillable coherence averagely available to Bob. It can be understood as a nonlocal advantage of quantum coherence, which has been attributed to the quantum correlations, such as quantum steering process [28, 38], Bell nonlocality [39], and quantum deficit-like correlations [40].

In the following section, we first verify the distribution inequality of the assisted distillable coherence numerically and experimentally. Then we intend to study the relation between the multipartite correlation and the assisted coherence distillation. As is known, the monogamy relation [41–43] was discovered as a fundamental perspective for revealing multipartite correlations. It directly motivates us to derive the monogamy relation in the assisted coherence distillation as in theorem 2.

We define the distribution core $\tau_A \equiv C_{CoP}^{A|BC} - C_{CoP}^{A|B} - C_{CoP}^{A|C}$, and the symmetrized form $\tau \equiv \min(\tau_A, \tau_B, \tau_C)$ with τ_B and τ_C corresponding to the cases that B, C acts as the auxiliary system, respectively. For the cases of pure state, when the state is in a completely product form $|\Psi_A\rangle \otimes |\Psi_B\rangle \otimes |\Psi_C\rangle$, or partially product state with bipartite correlations, i.e., $|\Psi_{AB}\rangle \otimes |\Psi_C\rangle$, $|\Psi_{AC}\rangle \otimes |\Psi_B\rangle$, $|\Psi_A\rangle \otimes |\Psi_{BC}\rangle$, there is $\tau = 0$. That is $\tau \neq 0$ implies that the state cannot be written as the product forms, which is equivalent to indicating the existence of multipartite correlations. For the cases $\tau > 0$, different values of τ reflect different monogamy degrees of the assisted distillable coherence, and which will be found to have relationship with the multipartite correlation. In the following section, several typical measures of multipartite correlations are compared with the results of τ by numerical calculations.



3. Experimental demonstration distribution of coordinately assisted distillable coherence

In order to prepare entangled photon pairs, the 405 nm pump laser (3 mW) outputs from the continuous laser. The 810 nm photon pairs are generated by spontaneous parametric down conversion of the 1.5 cm-long type-II periodically poled potassium titanyl phosphate (PPKTP) nonlinear crystal in Sagnac loop (shown in the module (a) in figure 1). The entangled state is encoded in the polarization modes, and thus the two-qubit space is spanned by the basis vectors $\{|i\rangle_A |j\rangle_B\}$ with $i, j = 0, 1$. We obtain 45 000 s entangled photon pairs with the concurrence being 0.982, and the fidelity to the maximally entangled pure state $(|11\rangle_{AB} + |00\rangle_{AB})/\sqrt{2}$, reaching 99.8%. In the module (b) of figure 1, by using the beam displacer (BD), the polarization modes of photon B ($|j\rangle_B$) is coupled to the spatial modes ($|k\rangle_C$ with $k = 0, 1$). Based on the polarization–spatial interactions, we prepare the tripartite states [36]. Moreover, in this work, two types of quantum channels are constructed to realize the polarization–spatial interactions, one is the partial controlled-Not (CN) gate channel corresponding to the following map [37]:

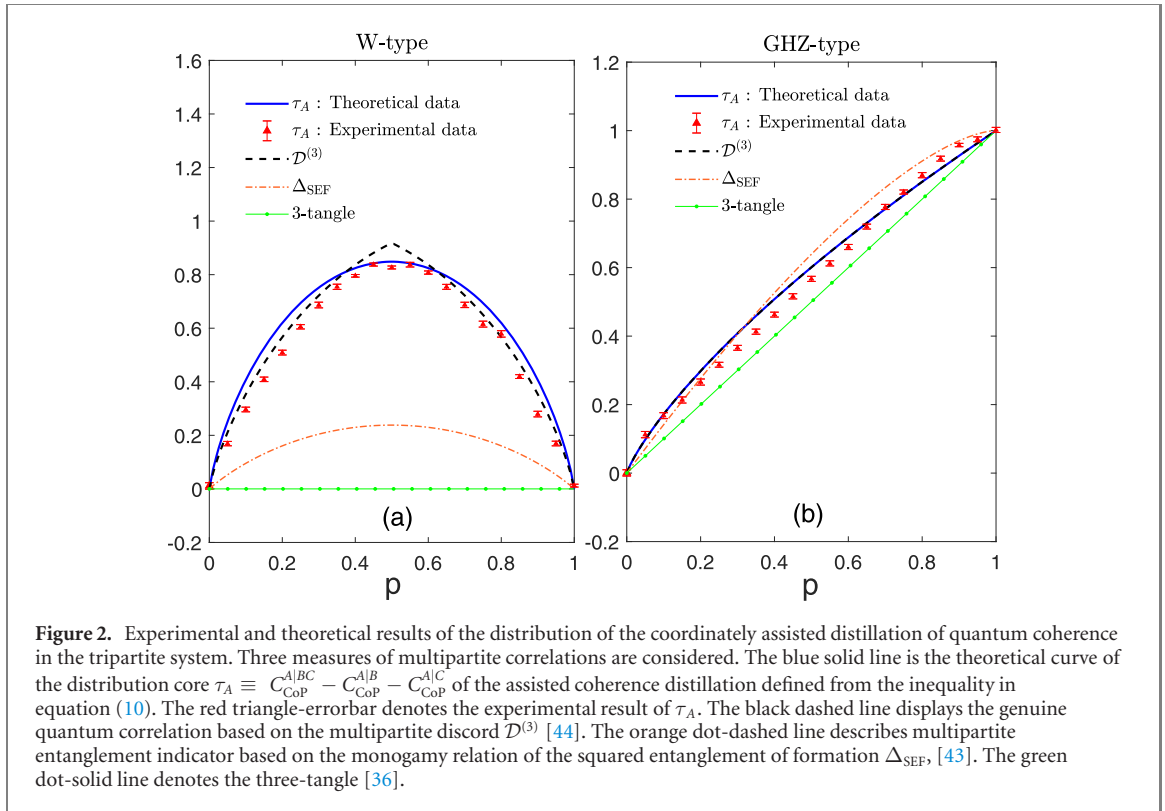
$$\begin{aligned} |0\rangle_B |0\rangle_C &\rightarrow |0\rangle_B |0\rangle_C, \\ |1\rangle_B |0\rangle_C &\rightarrow \sqrt{1-p}|1\rangle_B |0\rangle_C + \sqrt{p}|1\rangle_B |1\rangle_C \end{aligned} \quad (13)$$

and the other is the conditional bit-flip (BF) channel [37]:

$$\begin{aligned} |0\rangle_B |0\rangle_C &\rightarrow |0\rangle_B |0\rangle_C, \\ |1\rangle_B |0\rangle_C &\rightarrow \sqrt{1-p}|1\rangle_B |0\rangle_C + \sqrt{p}|0\rangle_B |1\rangle_C. \end{aligned} \quad (14)$$

With the help of the two channels above, we prepare two types of three-qubit entangled states [36]. For the initial state $\frac{1}{\sqrt{3}}(|10\rangle_{AB} + \sqrt{2}|01\rangle_{AB})|0\rangle_C$, the CN channel produces the W -type state

$$|\phi\rangle = \frac{1}{\sqrt{3}}|100\rangle + \sqrt{\frac{2}{3}}(\sqrt{1-p}|010\rangle + \sqrt{p}|001\rangle). \quad (15)$$



For $p = 1/2$, the state becomes the W state. The subscripts A, B, C are omitted for simplicity. In the experiment, the parameter p can be simulated by the rotation angle θ of HWP₁ with the relation $p = \sin^2(2\theta)$.

For the initial state $\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})|0\rangle_C$, the BF channel produces the GHZ-type state

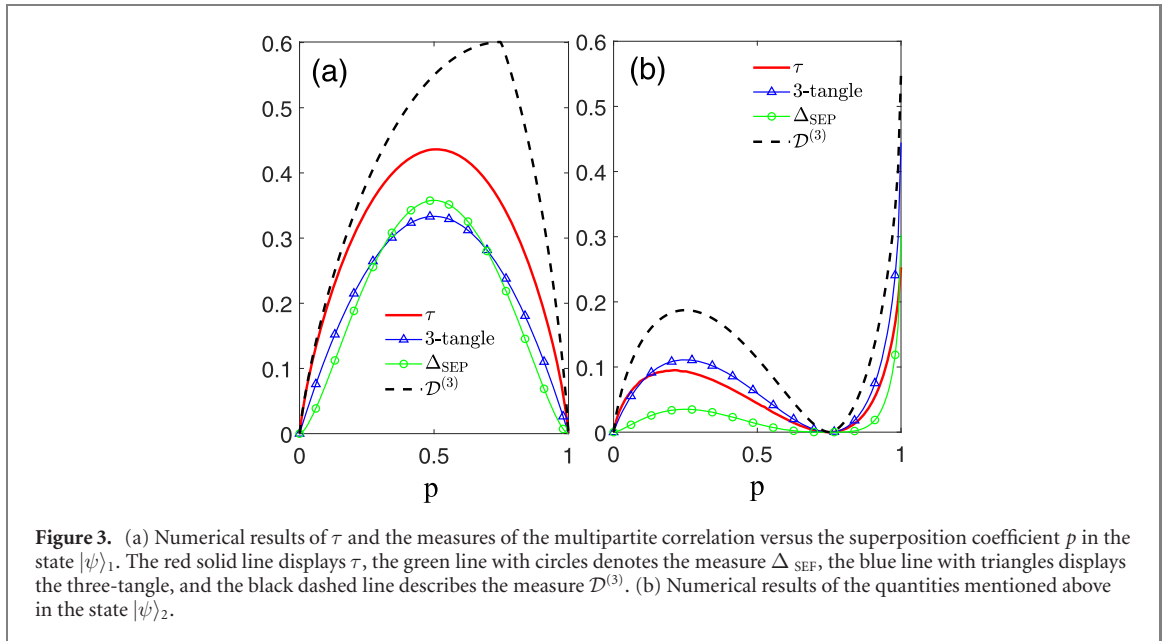
$$|\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + \sqrt{1-p}|110\rangle + \sqrt{p}|111\rangle), \quad (16)$$

which becomes the GHZ state for $p = 1$. In the following section, we experimentally test the measure to evaluate the assisted distillable coherence and verify the inequalities (10) based on the prepared tripartite entangled states.

Experimental results. In the experiment, we perform optimal projective measurements on Alice to obtain the assisted distillable coherence $C_{\text{CoP}}^{A|BC}$ in equation (5). One can find that the optimal measurement basis should be $(|0\rangle \pm |1\rangle)/\sqrt{2}$, which is due to that both GHZ-type and W -type states satisfy the Schmidt decomposition in equation (8). The tomography is performed on the residual state ρ_{BC}^{\pm} (corresponding to the probability of the measurements on Alice) to evaluate the assisted distillable coherence of BC .

To obtain $C_{\text{CoP}}^{A|B}$ and $C_{\text{CoP}}^{A|C}$, one should take into account the reduced density ρ_{AB} and ρ_{AC} . In order to find the optimal measurement on Alice, we introduce a general set of projective measurement basis denoted by $\cos\theta|0\rangle \pm \sin\theta e^{i\varphi}|1\rangle$. First, let us study the W -type state, since the expression of the relative entropy is complicate, another measure, i.e., l_1 norm of coherence [13], is employed. By numerically calculation, we find the behavior of the l_1 norm of coherence is similar with the relative entropy of coherence in W -type state. One can easily obtain the average l_1 norm of coherence of the subsystems B and C and which is found to be proportional to $\sqrt{1-p} \sin\theta \cos\theta$. Obviously, the measurement of $\theta = \pi/4$ (i.e., the measurement basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$) is optimal to help the system B (C) to capture the maximal average coherence. For the GHZ-type state, the relative entropy of coherence is easy to obtained (details are shown in the appendix). We find that the optimal measurement basis of A are $(|0\rangle \pm |1\rangle)/\sqrt{2}$ to obtain $C_{\text{CoP}}^{A|B}$, while $(|0\rangle, |1\rangle)$ to obtain $C_{\text{CoP}}^{A|C}$.

In figure 2(a), we prepare the W -type tripartite state, and perform the optimal measurement on photon A . Then the residual states of the subsystem BC , B , and C can be detected by tomography. Furthermore, one obtains the coherence of the residual states, and thus the assisted distillable coherence, i.e., $C_{\text{CoP}}^{A|BC}$, $C_{\text{CoP}}^{A|B}$, and $C_{\text{CoP}}^{A|C}$. For experimental simplicity, we only investigate $\tau_A \equiv C_{\text{CoP}}^{A|BC} - C_{\text{CoP}}^{A|B} - C_{\text{CoP}}^{A|C}$ and show its theoretical and experimental results versus the superposition parameter p in figure 2(a). One can find that $\tau_A \geq 0$ in the whole parameter region, which verifies the inequality (10). Moreover, τ_A reaches its maximum at



$p = 1/2$, where the tripartite state becomes the W state, i.e., $|\phi\rangle_W = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. While, τ_A reaches zero at $p = 0$ and 1, where the tripartite quantum correlations degenerate into the bipartite correlations figure 2(a).

It is known that W -type state is rich in genuine tripartite quantum correlations [43, 44]. We believe that the values of τ_A , which reflects the degrees of the monogamous distribution of the assisted distillable coherence, are in close relationship with the genuine quantum correlations. We numerically calculate the genuine tripartite quantum entanglement Δ_{SEP} [43] and the genuine tripartite quantum discord $\mathcal{D}^{(3)}$ [44] (the definitions can be found in the appendix). One can find that τ_A , Δ_{SEP} , and $\mathcal{D}^{(3)}$ reach the zero values for the same parameters $p = 0, 1$, and reach their respective maximal values at the same position of $p = 1/2$. The increase (decrease) of τ_A is synchronized with the increase (decrease) of Δ_{SEP} and $\mathcal{D}^{(3)}$. We also consider another well-known measure, i.e., the three-tangle [36], which is found to be always zero in the considered region of p . It implies that nonzero τ_A should be connected with the multipartite correlation that cannot be detected by three-tangle but can be characterized by Δ_{SEP} and $\mathcal{D}^{(3)}$.

In figure 2(b), the case of GHZ-type states is studied. One can see that the four quantities τ_A , Δ_{SEP} , $\mathcal{D}^{(3)}$, and three-tangle all increase monotonously as p increases, which is different from that in the case of W -type states. More specially, τ_A and $\mathcal{D}^{(3)}$ are completely coincident. The zero values of the four quantities are found at $p = 0$, where the genuine tripartite correlation disappears, instead, only bipartite correlation exists. While, at $p = 1$, the state becomes GHZ state, which displays the maximal genuine tripartite correlation.

Other types of states with multipartite correlations are also considered and some of the numerical results are shown in the appendix. All the results show the nearly synchronous increase and decrease of the quantity τ and the multipartite correlation measure.

4. Conclusion

We have considered the issue of assisted coherence distillation in the asymptotic limit. Different types of measurements on the auxiliary system were discussed. Then, we focused on coordinately performing identical projective measurements on the auxiliary of each resource state copy. In this coordinate asymptotic scenario, the coherence distillation process branches into many subprocesses, each of which has a corresponding distillation rate. Finally, a simple measure of the assisted distillable coherence is obtained as the maximal average coherence of the residual states with the maximum being taken over all the projective measurements on the auxiliary. More importantly, the measure is applicable for the cases that the considered system and its auxiliary are in a composite mixed state. In addition, it is convenient to be experimentally tested in one-copy state to evaluate the assisted distillable coherence available in the coordinate asymptotic scenario.

We for the first time investigated the assisted coherence distillation in multipartite systems. Monogamy-like inequalities were given to constrain the distribution of the assisted distillable coherence in the subsystems. We experimentally prepared two types of tripartite correlated states, i.e., the W -type and

GHZ-type states, and experimentally test the measure derived by us to evaluate the assisted distillable coherence. Theoretical and experimental results agree well to verify the distribution inequality. We find that the monogamy-like law of assisted coherence distillation can be used to detect the existence of multipartite correlations. Three measures of multipartite correlation were considered. The numerical results reveal the close relationship between the assisted coherence distillation and the multipartite quantum correlations.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

Appendix A. Proof of lemma 1

In this appendix, we will show the proof of lemma 1. Based on our proposed CoLQICC, after coordinately performing the projective measurements on the n copies of the resource state, we obtain a mixed form of Alice's post-measurement state and Bob's residual state:

$$\Lambda_{A:\text{CoP}}(\rho_{AB}^{\otimes n}) = \sum_i P_i (\Xi_A^i \otimes \rho_B^i)^{\otimes n}, \quad (17)$$

with $P_i = \text{Tr}(\Xi_A^i \otimes \mathbb{I}_B \rho_{AB})$, and $\rho_B^i = \text{Tr}_A(\Xi_A^i \otimes \mathbb{I}_B \rho_{AB})/P_i$. Recalling the map $\Lambda_{\text{CoP}} \equiv \Lambda_B^{\text{IO}} \circ \Lambda_{A:\text{CoP}}$ and substituting the equation above into the definition of the coordinately assisted coherence distillation, we have:

$$\begin{aligned} C_{\text{CoP}}^{A|B}(\rho) &= \sup\{\mathcal{R} : \liminf_{n \rightarrow \infty} \inf_{\Lambda_{\text{CoP}}} \|\Lambda_{\text{CoP}}(\rho_{AB}^{\otimes n}) - \Phi_2^{\otimes \lfloor n\mathcal{R} \rfloor}\|_1 = 0\} \\ &= \max_{\Xi_A^i} \sup\left\{\mathcal{R} : \liminf_{n \rightarrow \infty} \inf_{\text{IO}_B} \left\| \sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - \Phi_2^{\otimes \lfloor n\mathcal{R} \rfloor} \right\|_1 = 0\right\}. \end{aligned} \quad (18)$$

We find that after the projective measurements, IOs are only performed on Bob's side to finally realize the goal of the coherence distillation. Therefore, focusing on the core part, the trace norm actually becomes:

$$\begin{aligned} D^{\Xi_A^i}(\rho_B^i) &\equiv \left\| \sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - \sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Phi_2^{\otimes \lfloor nR_i \rfloor} \right\|_1 \\ &\leq \sum_i P_i \left\| (\Xi_A^i)^{\otimes n} \otimes \Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - (\Xi_A^i)^{\otimes n} \otimes \Phi_2^{\otimes \lfloor nR_i \rfloor} \right\|_1 \\ &= \sum_i P_i \left\| \Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - (\Phi_2)^{\otimes \lfloor nR_i \rfloor} \right\|_1, \end{aligned} \quad (19)$$

where the first inequality is due to the convexity of trace norm, and the second equality comes from the fact $\|\Xi \otimes M\|_1 = \|M\|_1$ for a Hermitian matrix M and a matrix Ξ of rank 1. Now recalling the original concept of coherence distillation in [20], when $n \rightarrow \infty$, proper IOs on Bob can be found to make $(\rho_B^i)^{\otimes n}$ approach $(\Phi_2)^{\otimes \lfloor nR_i \rfloor}$ asymptotically, i.e., existing an arbitrarily small $\varepsilon_i \rightarrow 0$ that the trace norm satisfies

$$\inf_{\text{IO}_B} \left\| \Lambda_B^{\text{IO}}(\rho_B^i)^{\otimes n} - (\Phi_2)^{\otimes \lfloor nR_i \rfloor} \right\|_1 \leq \varepsilon_i. \quad (20)$$

Then one has

$$\liminf_{n \rightarrow \infty} \inf_{\text{IO}_B} D^{\Xi_A^i}(\rho_B^i) \leq \varepsilon \equiv \lim_{n \rightarrow \infty} \sum_i P_i \varepsilon_i \rightarrow 0, \quad (21)$$

which implies that the process of the coordinately assisted coherence distillation, i.e., the asymptotic incoherent transformation $\rho_{AB}^{\otimes n} \xrightarrow{\text{CoLQICC}^{1-\varepsilon}} \Phi_2^{\otimes \lfloor nR \rfloor}$ is achievable as $n \rightarrow \infty$, $\varepsilon \rightarrow 0$. Subsequently, the rate of coherence distillation in this assisted scenario is a probabilistic sum of all the parts: $R = \sum_i P_i R_i$, whose

maximum is taken over all the projective measurements $\{\Xi_A^i\}$. Finally, the rate of coordinately assisted distillation of coherence becomes $\mathcal{R} = \max_{\Xi_A^i} \sum_i P_i R_i$.

Appendix B. Proof of theorem 1

First let us prove the upper bound of the rate of the coordinately assisted coherence distillation, i.e., $\mathcal{R} \leq \max_{\{\Xi_A^i\}} \sum_i P_i C_r(\rho_B^i)$.

Due to the continuity of the entropy, for two states ρ_{AB}, σ_{AB} , supposing the trace norm satisfies $\|\rho_{AB} - \sigma_{AB}\|_1 \leq \varepsilon$ (with $0 \leq \varepsilon \leq 1/2$), the QI relative entropy (i.e., the relative entropy between a state and a QI state) is proved to be continuous [24], i.e.,

$$|C_r^{A|B}(\rho_{AB}) - C_r^{A|B}(\sigma_{AB})| \leq \varepsilon \log_2 d_{AB} + 2h(\varepsilon/2), \quad (22)$$

where the QI relative entropy $C_r^{A|B}(\rho_{AB}) \equiv S(\Delta^B \rho_{AB}) - S(\rho_{AB})$ with $S(\rho)$ being the von Neumann entropy, and the function $h(x) \equiv -x \log_2(x) - (1-x) \log_2(1-x)$, and d_{AB} is the dimension of the Hilbert space. For a projective measurement $\{\Xi_A^i\}$ on Alice, when we have the trace

norm $\|\sum_i P_i \Lambda_B^{IO}(\Xi_A^i \otimes \rho_B^i)^{\otimes n} - \sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Phi_2^{\otimes n R_i}\|_1 \leq \varepsilon$ at the limit of $n \rightarrow \infty$ and taking the infimum over Λ_B^{IO} , one can obtain the asymptotic continuity

$$C_r^{A|B} \left[\sum_i P_i \Lambda_B^{IO}(\Xi_A^i \otimes \rho_B^i)^{\otimes n} \right] \geq C_r^{A|B} \left[\sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Phi_2^{\otimes n R_i} \right] - f(\varepsilon), \quad (23)$$

where $f(\varepsilon) \equiv n \varepsilon \log_2 d_{AB} + 2h(\varepsilon/2)$. The right-hand side (rhs) of the inequality

$$\begin{aligned} \text{RHS} &= S \left[\sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Delta \Phi_2^{\otimes n R_i} \right] - S \left[\sum_i P_i (\Xi_A^i)^{\otimes n} \otimes \Phi_2^{\otimes n R_i} \right] - f(\varepsilon) \\ &= \sum_i P_i S(\Delta \Phi_2^{\otimes n R_i}) + H\{P_i\} - \sum_i P_i S(\Phi_2^{\otimes n R_i}) - H\{P_i\} - f(\varepsilon) \\ &= \sum_i n P_i R_i S(\Delta \Phi_2) - f(\varepsilon) \\ &= n \left[\sum_i P_i R_i - \frac{1}{n} f(\varepsilon) \right], \end{aligned} \quad (24)$$

In the second equality, we make use of the property of von Neumann entropy, i.e., $S[\sum_i p_i |i\rangle\langle i| \otimes \rho_i] = H(p_i) + \sum_i p_i S(\rho_i)$ where $H(p_i)$ is the Shannon entropy and $|i\rangle$ are orthogonal states. When $n \rightarrow \infty$, there is $\varepsilon \rightarrow 0$. In addition, since the relative entropy cannot be increased by the IOs, one has

$$C_r^{A|B} \left[\sum_i P_i (\Xi_A^i \otimes \rho_B^i)^{\otimes n} \right] \geq C_r^{A|B} \left[\sum_i P_i \Lambda_B^{IO}(\Xi_A^i \otimes \rho_B^i)^{\otimes n} \right] \geq n \sum_i P_i R_i. \quad (25)$$

Based on the definition of the relative entropy in terms of entropy, one can easily have $C_r^{A|B}[\sum_i P_i (\Xi_A^i \otimes \rho_B^i)^{\otimes n}] = n \sum_i P_i C_r(\rho_B^i)$. Thus, the upper bound is given in the inequality

$$R = \sum_i P_i R_i \leq \sum_i P_i C_r(\rho_B^i). \quad (26)$$

Then one can obtain the maximum of R by taking all the projective measurement $\{\Xi_A^i\}$, i.e., $\mathcal{R} = \max_{\{\Xi_A^i\}} \sum_i P_i R_i \leq \max_{\{\Xi_A^i\}} \sum_i P_i C_r(\rho_B^i)$.

Now, we should prove that the upper bound of the distillation rate can be achieved. The typicality technique will be employed to analyze the asymptotic limit case [20, 45]. Let us start from the purification of ρ_{AB} , i.e., $\rho_{AB} \xrightarrow{\text{purification}} |\Psi_{ABE}\rangle\langle\Psi_{ABE}|$. We suppose that the optimal projective measurement performed on system A is $\{\Pi_A^\nu\}$, and Alice sends the outcomes to Bob by a classical way. Then the post-measurement state, corresponding to the projector $\Pi_A^\nu \equiv |\Pi_A^\nu\rangle\langle\Pi_A^\nu|$, becomes proportional to $|\Pi_A^\nu\rangle\langle\Pi_A^\nu| \psi_{BE}^\nu$ with the probability $P_\nu = \text{Tr}(\Pi_A^\nu \otimes \mathbb{I}_{BE} \rho_{ABE})$. Then, to the n copies of $|\Psi_{ABE}\rangle$, after coordinately and independently performing the projective measurement $(\Pi_A^\nu)^{\otimes n}$, the post-measurement state will be proportional to

$$|\Phi_{ABE}^\nu\rangle^{\otimes n} \sim |\Pi_A^\nu\rangle^{\otimes n} |\psi_{BE}^\nu\rangle^{\otimes n}. \quad (27)$$

Then, if we implement the type measurement M_P on the subsystem B , i.e.,

$$M_P = \sum_{i^n \in T_n^{B,\nu}(P)} |i^n\rangle\langle i^n|, \tag{28}$$

where $|i^n\rangle \equiv |i_1, i_2, \dots, i_n\rangle_\nu$ describes the typical state sequence corresponding to the space of the post-measurement states. Each group $\{|i_n\rangle\}$, corresponding to the n th copy, can be the reference basis on which the quantum coherence is defined. The type measurement M_P , consisting of the projectors, can help us to choose all the typical sequences corresponding to the probability distribution P , which derives from the considered state. Thus, P is used to represent the type of strings $|i^n\rangle$ with length n . $T_n^{B,\nu}(P)$ denotes the type class of P corresponding to the measurement Π_A^ν , then δ -typical ($\delta > 0$) class satisfies

$$c = \left\{ |i^n\rangle : \left| -\frac{1}{n} \log p_{i^n} - H(P) \right| < \delta \right\}, \tag{29}$$

where the probability sequence $p_{i^n} = p_{i_1}^\nu p_{i_2}^\nu \dots p_{i_n}^\nu$, with the definition $p_i = \langle i | \rho_B^\nu | i \rangle$ for each set of basis $\{i_n\}$ and ρ_B^ν being the reduced density matrix of system B after the measurement Π_A^ν . The Shannon entropy $H(P) = -\sum_i p_i \log p_i$. The length of the δ -typical class $|T_n^{B,\nu}(P)|$ should be

$$2^{n(S(\Delta\rho_B^\nu)-\delta)} \leq |T_n^{B,\nu}(P)| \leq 2^{n(S(\Delta\rho_B^\nu)+\delta)}, \tag{30}$$

i.e., $|T_n^{B,\nu}(P)|$ indicates the number of the typical sequences, and $\Delta(\rho_B^\nu) = \sum_i |i\rangle\langle i| \rho_B^\nu |i\rangle\langle i|$ with $|i\rangle$ being the reference basis vector in each copy, on which the quantum coherence is defined. Then the dimension of the typical space holds

$$2^{n(S(\Delta\rho_B^\nu)-\delta)} \leq \dim[T_n^{B,\nu}(P)] \leq 2^{n(S(\Delta\rho_B^\nu)+\delta)}. \tag{31}$$

After the measurement Π_A^ν and the type measurement, the state of B and E can be expressed as

$$|\Phi_{BE}^\nu\rangle_{T_n}^{\otimes n} = \frac{1}{\sqrt{|T_n^{B,\nu}(P)|}} \sum_{i^n \in T_n^{B,\nu}(P)} |i^n\rangle |\varphi_{E,\nu}^{i^n}\rangle. \tag{32}$$

Due to the typical subspace theorem [45] and the fact $|T_n^{B,\nu}(P)| = |F_\nu| \cdot |M_\nu|$, there is a partition of the type class $T_n^{B,\nu}(P)$ into subsets $\{f\}$ with the length $|F_\nu|$, and the total number of the subsets is $|M_\nu|$, and each of the subset can be denoted by the vectors of $\{m\}$.

Since the property of the entropy

$$\begin{aligned} S(\Delta\rho_B^\nu) &= S[\Delta^B(|\psi_{BE}^\nu\rangle\langle\psi_{BE}^\nu|)] \\ &= I_{B:E}[\Delta^B(|\psi_{BE}^\nu\rangle\langle\psi_{BE}^\nu|)] + S_{B|E}[\Delta^B(|\psi_{BE}^\nu\rangle\langle\psi_{BE}^\nu|)], \end{aligned} \tag{33}$$

where the post-measurement state $|\psi_{BE}^\nu\rangle$ comes from equation (27), and $I_{B:E}$ denotes the mutual information and $S_{B|E}$ is the conditional entropy. Then the length of the set $\{m\}$ is $|M_\nu| \approx 2^{nI_{B:E}}$.

By using the Schmidt decomposition form of $|\psi_{BE}^\nu\rangle$, one can simply obtain

$$\Delta^B(|\psi_{BE}^\nu\rangle\langle\psi_{BE}^\nu|) = \sum_i q_i^\nu |i\rangle_B \langle i| \otimes |\varphi_i^\nu\rangle_E \langle \varphi_i^\nu|, \tag{34}$$

where $q_i^\nu = \sum_k (\lambda_k^\nu)^2 |\langle i|\phi_k\rangle|^2$ with λ_k^ν being the Schmidt coefficient and $|\phi_k\rangle$ is the Schmidt basis of B . Then the mutual information

$$\begin{aligned} I_{B:E}[\Delta^B(|\psi_{BE}^\nu\rangle\langle\psi_{BE}^\nu|)] &= S\left(\sum_i q_i^\nu |i\rangle\langle i|\right) + S\left(\sum_i q_i^\nu |\varphi_i^\nu\rangle\langle\varphi_i^\nu|\right) - S\left(\sum_i q_i^\nu |i\rangle\langle i| \otimes |\varphi_i^\nu\rangle\langle\varphi_i^\nu|\right) \\ &= S(\rho_E^\nu) = S(\rho_B^\nu). \end{aligned} \tag{35}$$

Thus, taking $\delta \rightarrow 0$ for simplicity, we have

$$\begin{aligned} |F_\nu| &= |T_n^{B,\nu}(P)|/|M_\nu| \\ &= 2^{n[S(\Delta\rho_B^\nu)-I_{B:E}]} \\ &= 2^{n[S(\Delta\rho_B^\nu)-S(\rho_B^\nu)]}. \end{aligned} \tag{36}$$

Let us relabel $(i^n) \rightarrow (f, m)$, then the post-measurement state can be expressed as:

$$\begin{aligned} |\Phi_{ABE}^{\nu}\rangle_{T_n}^{\otimes n} &= \sqrt{P_{\nu}} |\Pi_A^{\nu}\rangle^{\otimes n} \frac{1}{\sqrt{|F_{\nu}| \cdot |M_{\nu}|}} \sum_{f \in F_{\nu}, m \in M_{\nu}} |f\rangle |m\rangle \otimes |\varphi_{E,v}^{fm}\rangle \\ &= \sqrt{P_{\nu}} |\Pi_A^{\nu}\rangle^{\otimes n} \frac{1}{\sqrt{|F_{\nu}|}} \sum_{f \in F_{\nu}} |f\rangle \frac{1}{\sqrt{|M_{\nu}|}} \sum_{m \in M_{\nu}} |m\rangle |\varphi_{E,v}^{fm}\rangle, \end{aligned} \quad (37)$$

where $P_{\nu} = \text{Tr}(\Pi_A^{\nu} \otimes \mathbb{I}_{BC} \rho_{ABC})$. When we define $|\phi\rangle_{\nu}^f \equiv \frac{1}{\sqrt{|M_{\nu}|}} \sum_{m \in M_{\nu}} |m\rangle |\varphi_{E,v}^{fm}\rangle$, based on Uhlmann's theorem [46, 47], there exists a unitary U_{ν}^f on E such that $(\mathbb{I}_m \otimes U_{\nu}^f) |\phi\rangle_{\nu}^f \approx |\phi\rangle_{\nu}^0$ for each state $|\phi\rangle_{\nu}^f$. It implies that we can construct the incoherence operation described by a group of Kraus operators $\{K_r^{\nu}\}$, satisfying $\sum_r K_r^{\nu \dagger} K_r^{\nu} = \mathbb{I}$, on each ensemble $\{P_{\nu}, |\Phi_{ABE}^{\nu}\rangle_{T_n}^{\otimes n}\}$ [20], i.e.,

$$K_r^{\nu} = \mathbb{I}_A^{\otimes n} \otimes \sum_{f \in F_{\nu}} |f\rangle \langle f| \otimes |0\rangle \langle r| U_{\nu}^f. \quad (38)$$

We obtain

$$K_r^{\nu} |\Phi_{ABE}^{\nu}\rangle_{T_n}^{\otimes n} \approx \sqrt{P_{\nu}} |\Pi_A^{\nu}\rangle^{\otimes n} \frac{1}{\sqrt{|F_{\nu}|}} \sum_{f \in F_{\nu}} |f\rangle \otimes |0\rangle \langle r| \phi_{\nu}^0. \quad (39)$$

Now we approximately obtain the maximal coherent state $|\Phi_B^{\nu}\rangle_{|F_{\nu}|} = \frac{1}{\sqrt{|F_{\nu}|}} \sum_{f \in F_{\nu}} |f\rangle$. With $n \rightarrow \infty$ and the majorization condition $\Delta(|\Phi_B^{\nu}\rangle_{|F_{\nu}|}) \prec \Delta(\Phi_2^{\otimes nR})$ [20], where $(\Phi_B^{\nu})_{|F_{\nu}|} \equiv |\Phi_B^{\nu}\rangle_{|F_{\nu}|} \langle \Phi_B^{\nu}|$, and Φ_2 being the density of the two-dimensional maximal coherent state, one has $(\Phi_B^{\nu})_{|F_{\nu}|} \xrightarrow{\text{IO}} \Phi_2^{\otimes nR}$. There should be an equality of the length of the typical sequences, i.e.,

$$|F_{\nu}| = 2^{n[S(\Delta \rho_B^{\nu}) - S(\rho_B^{\nu})]} = 2^{nR_{\nu} S(\Delta \Phi_2)}, \quad (40)$$

and thus

$$R_{\nu} = [S(\Delta \rho_B^{\nu}) - S(\rho_B^{\nu})] = C_r(\rho_B^{\nu}), \quad (41)$$

which means that with the assistance of the optimal coordinate measurement Π_A^{ν} we can distillate Φ_2 by rate $R_{\nu} = C_r(\rho_B^{\nu})$ at the asymptotic limit. Finally, we have the total distillation rate $\mathcal{R} = \sum_{\nu} P_{\nu} C_r(\rho_B^{\nu}) = \max_{\{\Xi_A^i\}} \sum_i P_i C_r(\rho_B^i)$. The proof is completed.

Appendix C. Proof of theorem 2

Recalling the theorem 2, i.e.,

$$C_{\text{CoP}}^{A|BC}(\Psi_{ABC}) \geq C_{\text{CoP}}^{A|B}(\Psi_{ABC}) + C_{\text{CoP}}^{A|C}(\Psi_{ABC}). \quad (42)$$

Let us give the detailed proof by defining the core: $\tau_A \equiv C_{\text{CoP}}^{A|BC}(\Psi_{ABC}) - C_{\text{CoP}}^{A|B}(\Psi_{ABC}) - C_{\text{CoP}}^{A|C}(\Psi_{ABC})$. For a general bipartite state, we have $C_{\text{CoP}}^{A|B}(\rho_{AB}) \leq C_d^{A|B}(\rho_{AB}) \leq C_r^{A|B}(\rho_{AB})$, and for the type of states in equation (8), we have $C_{\text{CoP}}^{A|BC}(\Psi_{ABC}) = S(\Delta \rho_{BC})$, then there is an inequality:

$$\tau_A \geq S(\Delta \rho_{BC}) - C_r^{A|B}(\rho_{AB}) - C_r^{A|C}(\rho_{AC}). \quad (43)$$

It is known that relative entropy will increase by adding a subsystem [48], i.e., $C_r^{A|BC}(\Psi_{ABC}) \geq C_r^{A|C}(\rho_{AC})$, thus

$$\tau_A \geq S(\Delta \rho_{BC}) - C_r^{A|B}(\rho_{AB}) - C_r^{A|B|C}(\Psi_{ABC}). \quad (44)$$

By using the conditional entropy $S_{C|AB}$, we have $C_r^{A|B}(\rho_{AB}) + C_r^{A|B|C}(\Psi_{ABC}) = S_{C|AB}(\Delta^C \Psi_{ABC}) + S(\Delta^B \rho_{AB})$. Since relative entropy cannot be increased by performing CPTP operations, thus we have $S(\Delta^C \rho_{ABC} \| \rho_{AB} \otimes \Delta^C \rho_C) \geq S(\Delta^{BC} \rho_{ABC} \| \Delta^B \rho_{AB} \otimes \Delta^C \rho_C)$. By expanding the relative entropy, one will obtain the relationship between the conditional entropy, i.e., $S_{C|AB}(\Delta^{BC} \rho_{ABC}) \geq S_{C|AB}(\Delta^C \rho_{ABC})$. Then the rhs of the inequality (44) holds:

$$\text{RHS} \geq S(\Delta\rho_{BC}) - S_{C|AB}(\Delta^{BC}\Psi_{ABC}) - S(\Delta^B\rho_{AB}) = 0, \quad (45)$$

which is due to

$$\begin{aligned} S_{C|AB}(\Delta^{BC}\Psi_{ABC}) + S(\Delta^B\rho_{AB}) &= S(\Delta^{BC}\Psi_{ABC}) - S(\Delta^B\rho_{AB}) + S(\Delta^B\rho_{AB}) \\ &= S(\Delta^{BC}\Psi_{ABC}) = S(\Delta\rho_{BC}). \end{aligned} \quad (46)$$

Finally, we have

$$\tau_A \geq 0. \quad (47)$$

Now let us extend the proof to the multipartite cases of $N > 3$. For a pure state $|\Psi\rangle_{AB_1\dots B_N}$, with the dimension of the auxiliary is $\dim(\mathcal{H}_A) = 2$, and its Schmidt decomposition can be presented as:

$$|\Psi\rangle_{AB_1\dots B_N} = \sqrt{\lambda_1}|\phi_A^1\rangle\sum_{\{i'_B\}}|i'_{B_1}\dots i'_{B_N}\rangle + \sqrt{\lambda_2}|\phi_A^2\rangle|i_{B_1}\dots i_{B_N}\rangle, \quad (48)$$

where $\{i'_B\}$ denotes the subset consisting of the reference basis different from $|i_{B_1}\dots i_{B_N}\rangle$, and the two states of auxiliary satisfy $\langle\phi_A^2|\phi_A^1\rangle = 0$, then we have $C_{\text{CoP}}^{A|B_1\dots B_N}(\Psi_{AB_1\dots B_N}) = S(\Delta^{B_1\dots B_N}\Psi_{AB_1\dots B_N})$ with the definition $\Psi_{AB_1\dots B_N} \equiv |\Psi_{AB_1\dots B_N}\rangle\langle\Psi_{AB_1\dots B_N}|$. By using the tripartite inequality (44) and (45), we have

$$C_{\text{CoP}}^{A|B_1\dots B_N}(\Psi_{AB_1\dots B_N}) \geq C_r^{A|B_1}(\rho_{AB_1}) + C_r^{AB_1|B_2\dots B_N}(\Psi_{AB_1\dots B_N}), \quad (49)$$

where $C_r^{AB_1|B_2\dots B_N}(\Psi_{AB_1\dots B_N}) = S(\Delta^{B_2\dots B_N}\Psi_{AB_1\dots B_N})$. Then the tripartite inequality is reused that

$$\begin{aligned} C_r^{AB_1|B_2\dots B_N}(\Psi_{AB_1\dots B_N}) &\geq C_r^{AB_1|B_2}(\rho_{AB_1B_2}) + C_r^{AB_1B_2|B_3\dots B_N}(\Psi_{AB_1\dots B_N}) \\ &\geq C_r^{A|B_2}(\rho_{AB_1B_2}) + C_r^{AB_1B_2|B_3\dots B_N}(\Psi_{AB_1\dots B_N}), \end{aligned} \quad (50)$$

where we use the property of the relative entropy $C_r^{AB_1|B_2}(\rho_{AB_1B_2}) \geq C_r^{A|B_2}(\rho_{AB_1B_2})$. Then

$$\begin{aligned} C_r^{AB_1B_2|B_3\dots B_N}(\Psi_{AB_1\dots B_N}) &\geq C_r^{AB_1B_2|B_3}(\rho_{AB_1B_2B_3}) + C_r^{AB_1B_2B_3|B_4\dots B_N}(\Psi_{AB_1\dots B_N}) \\ &\geq C_r^{A|B_3}(\rho_{AB_1B_2}) + C_r^{AB_1B_2B_3|B_4\dots B_N}(\Psi_{AB_1\dots B_N}). \end{aligned} \quad (51)$$

By repeatedly using the inequalities above, one will finally have

$$\begin{aligned} C_{\text{CoP}}^{A|B_1\dots B_N}(\Psi_{AB_1\dots B_N}) &\geq \sum_{\alpha=1}^N C_r^{A|B_\alpha}(\rho_{AB_1B_2\dots B_N}) \\ &\geq \sum_{\alpha=1}^N C_{\text{CoP}}^{A|B_\alpha}(\rho_{AB_1B_2\dots B_N}). \end{aligned} \quad (52)$$

Then the proof is completed.

Appendix D. Proof of theorem 3

For a multipartite state $\rho_{AB_1B_2\dots B_N}$, a set of projective measurements $\{\Xi_A^i\}$ are performed on the subsystem A and classical communications are allowed among the subsystems, then the residual states of each subsystem B_j is $\rho_{B_j}^i$. Assuming that an optimal set of operations $\{\tilde{\Xi}_A^i\}$ help us to achieve the maximal average of the coherence, i.e., $\sum_i \tilde{P}_i \left[\sum_{j=1}^N C_r(\tilde{\rho}_{B_j}^i) \right]$. Because of the super additivity of coherence relative entropy, i.e., $C_r(\rho_1) + C_r(\rho_2) \leq C_r(\rho_{12})$, where the reduced density $\rho_{1(2)} = \text{Tr}_{2(1)}(\rho_{12})$. One has

$$\sum_i \tilde{P}_i \left[\sum_{j=1}^N C_r(\tilde{\rho}_{B_j}^i) \right] \leq \sum_i \tilde{P}_i C_r(\tilde{\rho}_{B_1B_2\dots B_N}^i). \quad (53)$$

Obviously, $\{\tilde{\Xi}_A^i\}$ is the optimal measurement to obtain the maximum of the subsystem coherence $\sum_{j=1}^N C_r(\rho_{B_j}^i)$ and not the coherence of the composite system $C_r(\rho_{B_1B_2\dots B_N}^i)$, thus when we take into account all the projective measurements $\{\Xi_A^i\}$, there is the following inequality, i.e.,

$$\begin{aligned} \sum_i \tilde{P}_i C_r(\tilde{\rho}_{B_1 B_2 \dots B_N}^i) &\leq \max_{\{\Xi_A^i\}} \sum_i P_i C_r(\rho_{B_1 B_2 \dots B_N}^i) \\ &= C_{\text{CoP}}^{A|B_1 B_2 \dots B_N}(\rho_{AB_1 B_2 \dots B_N}). \end{aligned} \tag{54}$$

The proof is completed.

Appendix E. Selection of optimal measurement

In this appendix, we show the details of how to choose the optimal measurement performed on subsystem A . The cases of GHZ-type and W -type states are considered. To obtain the assisted coherence distillation $C_{\text{CoP}}^{A|BC}$, the optimal measurement basis should be $(|0\rangle \pm |1\rangle)/\sqrt{2}$, which is due to that both GHZ-type and W -type states satisfy the Schmidt decomposition in equation (8). While, to obtain $C_{\text{CoP}}^{A|B}$ and $C_{\text{CoP}}^{A|C}$, one should take into account the reduced density ρ_{AB} and ρ_{AC} .

First, let us consider the case of W -type state. For the reduced state ρ_{AB} , we perform a general projective measurement, with the basis $|\varphi_+\rangle = \cos\theta|0\rangle + \sin\theta e^{i\varphi}|1\rangle$ and $|\varphi_-\rangle = \sin\theta|0\rangle - \cos\theta e^{i\varphi}|1\rangle$, on subsystem A . Then the corresponding probability are $P_+ = (1 + \cos^2\theta)/3$ and $P_- = (1 + \sin^2\theta)/3$, and the residual state of system B is (the classical communications between A and B are followed):

$$\begin{aligned} \rho_{+,B} &= \frac{3}{1 + \cos^2\theta} \left[\left(\frac{2\cos^2\theta}{3} p + \frac{\sin^2\theta}{3} \right) |0\rangle\langle 0| + \frac{2}{3}(1-p)|1\rangle\langle 1| \right. \\ &\quad \left. + \sin\theta \cos\theta e^{-i\varphi} \frac{\sqrt{2}}{3} \sqrt{1-p}|0\rangle\langle 1| + \sin\theta \cos\theta e^{i\varphi} \frac{\sqrt{2}}{3} \sqrt{1-p}|1\rangle\langle 0| \right]. \end{aligned} \tag{55}$$

Then, we make use of l_1 norm (defined as $C_{l_1} = \sum_{i \neq j} |\rho_{ij}|$, with ρ_{ij} being the off-diagonal elements) to measure the quantum coherence. Through numerical calculation, we find that the behavior of the l_1 norm of coherence is similar with the relative entropy of coherence, and the former is easy to calculate. For the residual state $\rho_{+,B}$ and $\rho_{-,B}$, we have that

$$C_{l_1}(\rho_{+,B}) \sim \sqrt{1-p} \frac{|\sin\theta \cos\theta|}{1 + \cos^2\theta}, \tag{56}$$

and which is same with $C_{l_1}(\rho_{-,B})$. Obviously, the average l_1 norm of coherence $\overline{C_{l_1}} = \sum_{i=+,-} P_i C_{l_1}(\rho_{iB})$ is proportional to $|\sin\theta \cos\theta|$, which means that $\overline{C_{l_1}}$ reaches its maximal value at $\theta = \pi/4$, i.e., the optimal measurement basis on A should be $(|0\rangle \pm |1\rangle)/\sqrt{2}$. The same is true for ρ_{AC} .

To the case of the GHZ-type state, we first consider the reduced state ρ_{AC} also by introducing a general form of the measurement basis $|\varphi_+\rangle = \cos\theta|0\rangle + \sin\theta e^{i\varphi}|1\rangle$ and $|\varphi_-\rangle = \sin\theta|0\rangle - \cos\theta e^{i\varphi}|1\rangle$. It is easy to obtain that $P_+ = P_- = 1/2$. After measurement, the residual states of C are $\rho_{+,C}$ and $\rho_{-,C}$:

$$\rho_{+,C} = \begin{pmatrix} 1-p \sin^2\theta & \sqrt{p(1-p)} \sin^2\theta \\ \sqrt{p(1-p)} \sin^2\theta & p \sin^2\theta \end{pmatrix}, \tag{57}$$

$$\rho_{-,C} = \begin{pmatrix} 1-p \cos^2\theta & p(1-p) \cos^2\theta \\ \sqrt{p(1-p)} \cos^2\theta & p \cos^2\theta \end{pmatrix}. \tag{58}$$

Then the eigenvalues of the two density are $(1 \pm \sqrt{1-p^2 \sin^2 2\theta})/2$, and finally we have the average relative entropy of coherence, i.e., the assisted distillable coherence of ρ_{AC} :

$$2C_{\text{CoP}}^{A|C}(\rho_{AC}) = \max_{\theta} F(p, \theta), \tag{59}$$

where

$$F(p, \theta) = H\{1-p \sin^2\theta, p \sin^2\theta\} + H\{1-p \cos^2\theta, p \cos^2\theta\} - 2H\left\{\frac{1}{2} \left(1 \pm \sqrt{1-p^2 \sin^2 2\theta} \right)\right\}, \tag{60}$$

with $H\{A, B\} \equiv -(A \log A + B \log B)$ is the binary Shannon entropy. By calculating the first and second order derivative of $F(p, \theta)$ with respect to θ , one can find that the minimum of $F(p, \theta)$ is at $\theta = \pi/4$, while the maximum can be reached at $\theta = 0$ or π , which implies that the best measurement basis are $\{|0\rangle, |1\rangle\}$, then $C_{\text{CoP}}^{A|C}(\rho_{AC}) = \frac{1}{2} H\{1-p, p\}$.

By doing a similar analysis to ρ_{AB} , the maximal value of $C_{\text{CoP}}^{A|B}(\rho_{AB})$ can be reached at $\theta = \frac{\pi}{4}$. Then the optimal measurement basis are $\{(|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2}\}$.

Appendix F. Measures of genuine tripartite quantum correlation Δ_{SEF} , and $\mathcal{D}^{(3)}$

In this appendix we introduce the concept of two types of genuine tripartite quantum correlation. The first one is based on the squared entanglement of formation, i.e., [43]

$$\Delta_{\text{SEF}}(\rho_{ABC}) = E_f^2(\rho_{A|BC}) - E_f^2(\rho_{A|B}) - E_f^2(\rho_{A|C}),$$

which detects that the multipartite entanglement is not stored in pairs of qubits. $E_f(\rho_{ij})$ is the entanglement of formation in the subsystem ρ_{ij} with the definition $E_f(\rho_{ij}) = \min_{\{p_m, |\phi_m^{ij}\rangle\}} \sum_m p_m S(\text{Tr}_i(|\phi_m^{ij}\rangle))$, where the minimum is taken over all the pure state decompositions $\{p_m, |\phi_m^{ij}\rangle\}$. In two-qubit quantum states, the entanglement of formation has an analytical expression $E_f(\rho_{ij}) = H\left\{\frac{1 \pm \sqrt{1 - C^2(\rho_{ij})}}{2}\right\}$, where

$H(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy and $C(\rho_{ij})$ is the concurrence of ρ_{ij} . Moreover, in a tripartite pure state $|\psi\rangle_{ABC}$, we have the relation $E_f^2(\rho_{A|BC}) = S^2(\rho_A)$ in which $E_f(A|BC)$ is the entanglement of formation in the partition $A|BC$ [43] and $S(\rho)$ is the von Neumann entropy.

Another concept is the multipartite discord with the definition (for the tripartite case) [44]:

$$\mathcal{D}^{(3)}(\rho) := \mathcal{D}(\rho) - \mathcal{D}^{(2)}(\rho), \quad (61)$$

where $\mathcal{D}^{(3)}(\rho)$ describes the genuine tripartite quantum correlation. Genuine correlations should contain all the contributions that cannot be accounted for considering any of the possible subsystems.

$\mathcal{D}(\rho) \equiv T(\rho) - \mathcal{J}(\rho)$ is called the total quantum discord with the total information (or correlation information) $T(\rho) \equiv S(\rho \| \rho_i \otimes \rho_j \otimes \rho_k)$, and the total classical correlation

$\mathcal{J}(\rho) \equiv \max_{P\{i,j,k\}} [S(\rho_i) - S(\rho_{ij}) + S(\rho_k) - S(\rho_{kij})]$ with the maximum among the six indices

permutations of the probability $P_{i,j,k} = P_{i|j,k} P_{j|k} P_k$. Note that $S(\rho_{ij}) \equiv \min_{\{E_i^j\}} S(i|\{E_i^j\})$ with respect to the

positive operator valued measure $\{E_i^j\}$, and the average entropy $S(i|\{E_i^j\}) = \sum_k p_k S(\rho_{i|E_m^j})$ with the

probability $p_k = \text{Tr}(E_i^j \otimes \mathbb{I} \rho_{ij})$ and the residual density $\rho_{i|E_m^j}$. Extending to the tripartite case, it becomes

$S(\rho_{kij}) \equiv \min_{\{E_i^j, E_j^i\}} S(k|\{E_i^j, E_j^i\})$. The minimum bipartite discord $\mathcal{D}^{(2)}(\rho)$ is defined as

$\mathcal{D}^{(2)}(\rho) \equiv \max[\mathcal{D}(\rho_{ij}), \mathcal{D}(\rho_{kj}), \mathcal{D}(\rho_{ki})]$. The symmetrized quantum discord

$\mathcal{D}(\rho_{ij}) \equiv \min[\mathcal{D}(\rho_{ij}), \mathcal{D}(\rho_{ji})]$, where $\mathcal{D}(\rho_{ij}) \equiv I(\rho_{ij}) - \max_{\{E_m^j\}} [S(\rho_i) - S(\rho_i|\{E_m^j\})]$ is the quantum

discord, and $I(\rho_{ij}) \equiv S(\rho_{ij} \| \rho_i \otimes \rho_j)$ is the mutual information. For the pure state $|\phi\rangle_{ijk}$, if the following inequality is satisfies: $I(\rho_{ij}) \geq I(\rho_{ik}) \geq I(\rho_{jk})$, there is a simple result that $\mathcal{D}^{(3)}(\rho) = S(\rho_k)$ [44]. Therefore,

for the GHZ-type states, it is easy to check by numerical calculation that $I(\rho_{AB}) \geq I(\rho_{AC}) \geq I(\rho_{BC})$. Then we have $\mathcal{D}_{\text{GHZ}}^{(3)} = S(\rho_C) = H\{1 - \frac{p}{2}, \frac{p}{2}\}$, and the minimum value $\min \mathcal{D}_{\text{GHZ}}^{(3)} = H\{1 - \frac{p}{2}, \frac{p}{2}\}|_{p=0} = 0$, while the

maximum value $\max \mathcal{D}_{\text{GHZ}}^{(3)} = H\{1 - \frac{p}{2}, \frac{p}{2}\}|_{p=1} = 1$.

As for the assisted distillable coherence, one can analytically obtain the minimum value of τ_A at $p = 0$, where the distillable coherence $C_{\text{CoP}}^{A|BC}(\rho_{ABC}) = 1$, $C_{\text{CoP}}^{A|C}(\rho_{ABC}) = 1$, and $C_{\text{CoP}}^{A|B}(\rho_{ABC}) = 0$, then $\tau_A = 0$. While,

at $p = 1$, we have $C_{\text{CoP}}^{A|BC}(\rho_{ABC}) = 1$, $C_{\text{CoP}}^{A|C}(\rho_{ABC}) = 0$, and $C_{\text{CoP}}^{A|B}(\rho_{ABC}) = 0$, which means $\tau_A = 1$. In

figure 2(b), the numerical and experimental results of τ_A show that in the case of the GHZ-type state, the behaviors of τ_A and $\mathcal{D}_{\text{GHZ}}^{(3)}$ are the same.

For the W -type states, the behavior of $\mathcal{D}^{(3)}$ is different from the GHZ-type states that when $0 \leq p \leq 0.5$, there are $I(\rho_{AB}) \geq I(\rho_{BC}) \geq I(\rho_{AC})$, then the tripartite quantum discord $\mathcal{D}^{(3)} = S(\rho_C)$. When $0.5 < p \leq 1$, there are $I(\rho_{AC}) \geq I(\rho_{BC}) \geq I(\rho_{AB})$, and thus $\mathcal{D}^{(3)} = S(\rho_B)$. Obviously, on both sides of the point $p = 0.5$, $\mathcal{D}^{(3)}$ behaves differently. We also discuss the relation between τ_A and $\mathcal{D}^{(3)}$. When $p = 0$ and 1, there will be $\mathcal{D}^{(3)} = 0$, where one can also find $\tau_A = 0$. While, at the special point of $p = 0.5$, the two quantities reach their maximum values $\mathcal{D}^{(3)} \simeq 0.918$ and $\tau_A \simeq 0.848$. More clearly, one can find the numerical and experimental results in figure 2(a), where τ_A and $\mathcal{D}^{(3)}$ display a similar behavior except the regions near the maximal value.

Appendix G. Numerical results of τ and the multipartite correlations in other types of three-qubit states

In this section, we numerically calculate the quantities of $\tau = \min(\tau_A, \tau_B, \tau_C)$ and the measure of multipartite correlation, i.e., three-tangle, Δ_{SEF} , and $\mathcal{D}^{(3)}$. We choose two tripartite pure states, which will show some different phenomena from those in figure 3. The states are

$$|\psi\rangle_1 = \sqrt{\frac{p}{3}}|000\rangle + \sqrt{\frac{p}{3}}|001\rangle + \sqrt{\frac{p}{3}}|011\rangle + \sqrt{1-p}|111\rangle, \quad (62)$$

$$|\psi\rangle_2 = \sqrt{\frac{p}{3}}|00+\rangle + \sqrt{\frac{p}{3}}|11-\rangle + \sqrt{\frac{p}{3}}|10+\rangle + \sqrt{1-p}|11-\rangle, \quad (63)$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. From the numerical results, we find that the minimum values of the four quantities appear at the same position, while the locations where the maximum values appear are slightly different. There are a roughly consistent increase and decrease process of τ and the measures of the multipartite correlation. We also check many other states, and all the results support the conclusion that larger values of τ correspond to stronger multipartite correlations.

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