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ORIGINAL PAPER

# On the monotonicity and convexity for generalized elliptic integral of the first kind

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**Abstract**
In this paper, we investigate the monotonicity and convexity of the function $x \mapsto \mathcal{K}_a(\sqrt{x})/\log(1+c/\sqrt{1-x})$ on $(0,1)$ for $(a,c) \in (0,1/2] \times (0,\infty)$, and the log-concavity of the function $x \mapsto (1-x)^{\lambda}\mathcal{K}_a(\sqrt{x})$ on $(0,1)$ for $\lambda \in \mathbb{R}$, where $\mathcal{K}_a(r)$ is the generalized elliptic integral of the first kind. These results are the generalization of [1, Theorem 2] and [2, Theorems 1.7 and 1.8], also give an affirmative answer of [3, Problem 3.1].

**Keywords** Gaussian hypergeometric function · Generalized elliptic integrals · Monotonicity · Log-concavity

**Mathematics Subject Classification** 33E05 · 26E60

## 1 Introduction

Throughout this paper, we denote $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a$, $b$ and $c$ with $-c \notin \mathbb{N}_0$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) \equiv {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad \text{for } |x| < 1,$$

where $(a)_0 = 1$ for $a \neq 0$ and $(a)_n$ is the *shifted factorial function* or *Pochhammer symbol* given by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $n \in \mathbb{N}$. Here $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ $(x > 0)$ is the Euler gamma function.

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The hypergeometric function $F(a, b; c; x)$ has a simple derivative formula

$$F'(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x). \tag{1.1}$$

The behavior of the hypergeometric function near $x = 1$ in the three cases $a + b < c$, $a + b = c$ and $a + b > c$ (see [4, Theorems 1.19 and 1.48]), is given by

$$\begin{cases} F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & a+b<c, \\ B(a, b)F(a, b; c; x) + \log(1-x) = R(a, b) + O((1-x)\log(1-x)), & a+b=c, \\ F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x), & a+b>c, \end{cases} \tag{1.2}$$

where $B(z, w) = [\Gamma(z)\Gamma(w)]/\Gamma(z+w)$ $(\Re(z) > 0, \Re(w) > 0)$ is the classical beta function,

$$R(a, b) = -2\gamma - \psi(a) - \psi(b) \tag{1.3}$$

is the Ramanujan constant, $\psi(z) = \Gamma'(z)/\Gamma(z)$ $(\Re(z) > 0)$ and $\gamma$ is the Euler-Mascheroni constant.

For $r \in (0, 1)$ and $a \in (0, 1)$, the generalized elliptic integrals of first and second kinds (see [5]) are defined by

$$\mathcal{K}_a \equiv \mathcal{K}_a(r) = \frac{\pi}{2} F\left(a, 1-a; 1; r^2\right), \tag{1.4}$$

$$\mathcal{E}_a \equiv \mathcal{E}_a(r) = \frac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right). \tag{1.5}$$

Clearly, $\mathcal{K}_a(0) = \mathcal{E}_a(0) = \pi/2$, $\mathcal{E}_a(1^-) = \sin(\pi a)/[2(1-a)]$ and $\mathcal{K}_a(1^-) = \infty$. In particular, $\mathcal{K}_{1/2}(r) \equiv \mathcal{K}(r)$ is the complete elliptic integral of the first kind. By symmetry of (1.4), we assume that $a \in (0, 1/2]$ in the sequel. In our parameter's setting, we denote by

$$R(a) \equiv R(a, 1-a) = -2\gamma - \psi(a) - \psi(1-a), \quad R(1/2) = 4\log 2, \tag{1.6}$$

where $R(a, b)$ is defined in (1.3).

It is well known that the generalized elliptic integral of the first kind plays an important role in different branches of modern mathematics such as classical real and complex analysis, quasiconformal mappings, number theory, such as the modulus of the plane Grötzsch ring [6–8] and Ramanujan's modular equation [9–17]. In particular, many remarkable results involving complete elliptic integral of the first kind can be found in [18–20] and its generalization (see [21–26]). For more informations and related recently published articles, we refer to the literature [27–37] and references therein.

As we know, the asymptotic formula for the zero-balanced case $a + b = c$ in (1.2) is due to Ramanujan (see [38]). In the case of $a = b = 1/2$, the Ramanujan asymptotic formula reduces to

$$\mathcal{K}(r) \sim \log \frac{4}{r'} \tag{1.7}$$

as $r \to 1^-$, where and in what follows $r' = \sqrt{1 - r^2}$.

In order to refine the asymptotic formula (1.7), Anderson, Vamanamurthy and Vuorinen [39] in 1992 conjectured and later Qiu et al. [40] proved that

$$\mathcal{K}(r) < \log\left(1 + \frac{4}{r'}\right) - \left(\log 5 - \frac{\pi}{2}\right)(1 - r) \tag{1.8}$$

holds for all $r \in (0, 1)$.

Recently, motivated by (1.8), Yang and Tian [1] investigated the monotonicity of the function $x \mapsto \mathcal{K}(\sqrt{x})/\log(1+4/\sqrt{1-x})$ on $(0, 1)$ and it is extended to the generalized elliptic integral of the first kind by Zhao et al. [3], where the authors proved that the function

$$r \mapsto \frac{\mathcal{K}_a(r)}{\log[1+2/(ar')]}$$

is strictly increasing on $(0, 1)$ if and only if $0.3199\cdots \le a \le 1/2$, and strictly decreasing on $(0, 1)$ if $0 < a \le 0.1899\cdots$. And they also pose two problems in the end of [3], one of which is stated as follows.

**Problem 1.1** *Determine the best possible function* $c(a)$ *for* $a \in (0, 1/2]$ *such that the function*

$$r \mapsto \frac{\mathcal{K}_a(r)}{\log[1+c(a)/r']}$$

*is strictly increasing or decreasing on* $(0, 1)$.

The first aim of this paper is to give an answer to Problem 1.1.

**Theorem 1.2** *Let* $a \in (0, 1/2]$, $c \in (0, \infty)$ *and define the function* $F(x)$ *on* $(0, 1)$ *by*

$$F(x) \equiv F(x, a, c) = \frac{\mathcal{K}_a(\sqrt{x})}{\log(1+c/\sqrt{1-x})}.$$

*Then we have the following conclusion:*

(1) $F(x)$ *is strictly increasing from* $(0, 1)$ *onto* $\big(\pi/[2\log(1+c)], \sin(a\pi)\big)$ *if and only if* $c \ge \max_{a\in(0,1/2]}\{c^*(a), e^{R(a)/2}\}$, *where* $c^*(a)$ *is the unique root of* $\varrho(c) = 1/[a(1-a)]$ *for* $c \in (0, \infty)$ *and* $\varrho(c)$ *is defined as in Lemma 2.6. In particular, the double inequality*

$$\frac{\pi}{2\log(1+c)} \log\left(1+\frac{c}{r'}\right) < \mathcal{K}_a(r) < \sin(a\pi)\log\left(1+\frac{c}{r'}\right) \tag{1.9}$$

*holds for all* $r \in (0, 1)$.

(2) *If* $0 < c \le \max_{a\in(0,1/2]}\{c_*(a), c_1(a)\}$, *then* $F(x)$ *is strictly decreasing from* $(0, 1)$ *onto* $\big(\sin(a\pi), \pi/[2\log(1+c)]\big)$, *where* $c_*(a)$ *is the unique root of* $\varrho(c) = R(a)$ *for* $c \in (0, \infty)$ *and* $c_1(a)$ *is given as in Lemma 2.5. In this case, the reverse inequality of* (1.9) *holds for all* $r \in (0, 1)$.

In the same paper as before, Yang and Tian [1] conjectured the function

$$x \mapsto (1-x)^p \mathcal{K}(\sqrt{x}) \tag{1.10}$$

is log-concave on $(0, 1)$ if and only if $p \ge 7/32$, and the function

$$x \mapsto \frac{\mathcal{K}(\sqrt{x})}{\log(1+4/\sqrt{1-x})} \tag{1.11}$$

is convex on $(0, 1)$. Very recently, these two conjectures had been proved by Wang et al. [2]. The remaining goal in this paper is to generalize the results of two functions in (1.10), (1.11) to the case of the generalized elliptic integral of the first kind. We will prove the following theorems.

**Theorem 1.3** *Let* $a \in (0, 1/2]$ *and define* $G(x)$ *on* $(0, 1)$ *by*

$$G(x) \equiv G(x, a, \lambda) = (1-x)^{\lambda} \mathcal{K}_a(\sqrt{x}).$$

*Then* $G(x)$ *is log-concave on* $(0, 1)$ *if and only if* $\lambda \ge \lambda(a) := [a(1-a)(a^2-a+2)]/2$ *and log-convex on* $(0, 1)$ *if and only if* $\lambda \le 0$.

**Theorem 1.4** *Let $a \in (0, 1/2]$, $c \in (0, \infty)$ and $F(x)$ be defined as in Theorem 1.2. Then $F(x)$ is convex on $(0, 1)$ if $c \geq \max_{a\in(0,1/2]} \left\{c^*(a), e^{R(a)/2}\right\}$, where $c^*(a)$ is defined as in Theorem 1.2.*

## 2 Lemmas

In order to prove our main results, we need to introduce some basic knowledges and establish several lemmas which we present in this section.

Let us recall the differentiation formulae for the generalized complete elliptic integrals, which can be found in [5, Theorem 4.1]

$$\frac{d\mathcal{K}_a}{dr} = \frac{2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{rr'^2}, \qquad \frac{d\mathcal{E}_a}{dr} = \frac{2(a-1)(\mathcal{K}_a - \mathcal{E}_a)}{r},$$
$$\frac{d}{dr}(\mathcal{K}_a - \mathcal{E}_a) = \frac{2(1-a)r\mathcal{E}_a}{r'^2}, \qquad \frac{d}{dr}(\mathcal{E}_a - r'^2\mathcal{K}_a) = 2ar\mathcal{K}_a.$$

It is worth noting that the L'Hôpital Monotone Rule has been widely applied, see [41–43]. In this paper, we also need another useful monotone rule to deal with the ratio of power series. Before stating this monotone rule, we need to introduce a useful auxiliary function $H_{f,g}$; see [44] for more properties. For $-\infty \leq a < b \leq \infty$, let $f$ and $g$ be differentiable on $(a, b)$ and $g' \neq 0$ on $(a, b)$. Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'}g - f.$$

In particular, if $f$ and $g$ are twice differentiable on $(0, 1)$, then we have

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2}\left(\frac{f'}{g'}g - f\right) = \frac{g'}{g^2}H_{f,g}, \tag{2.1}$$

$$H'_{f,g} = \left(\frac{f'}{g'}\right)' g. \tag{2.2}$$

**Lemma 2.1** ( [45]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \mathbb{N}_0$. Then the following statements hold true:*

(1) *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing) for all $n \geq 0$, then $f(x)/g(x)$ is strictly increasing (decreasing) on $(0, r)$;*
(2) *If for certain $m \in \mathbb{N}$ the sequence $\{a_k/b_k\}_{0\leq k\leq m}$ and $\{a_k/b_k\}_{k\geq m}$ both are non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively. Then $f(x)/g(x)$ is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \geq (\leq)0$. If $H_{f,g}(r^-) < (>)0$, then there exists $x_0 \in (0, r)$ such that $f(x)/g(x)$ is strictly increasing (decreasing) on $(0, x_0)$ and strictly decreasing (increasing) on $(x_0, r)$.*

***Remark 2.1*** The first part of Lemma 2.1 is first established by Biernacki and Krzyz [46], while the second part comes from Yang et al. [47, Theorem 2.1]. But we cite the latest version of the second part [45, Lemma 2], where the authors have corrected the previous bug [47, Theorem 2.1].

**Lemma 2.2** (1) *The function $r \mapsto (\mathcal{E}_a - r'^2\mathcal{K}_a)/(r^2\mathcal{K}_a)$ is strictly decreasing from $(0, 1)$ onto $(0, a)$.*

(2) *The function* $r \mapsto (\mathcal{E}_a - r'^2\mathcal{K}_a)/r^2$ *has positive Maclaurin coefficients and maps* $(0, 1)$ *onto* $(\pi a/2, \sin(\pi a)/[2(1-a)])$.

***Proof*** Parts (1) and (2) can be found in the literature [4, Theorem 4.22 (viii) and (ix)]. □

It is still not easy to study the monotonicity of the ratio of functions related to the square of power series even by using Lemma 2.1. The following proposition is the key tool to deal with the square of power series. We will give the following recurrence formula for the Maclaurin coefficients of $F(a, b; c; x)^2$, although we only use a special case in this paper.

**Proposition 2.3** *Let* $a, b, c \in \mathbb{R}$ *with* $\{-c, -2c\} \cap \mathbb{N}_0 = \emptyset$, *and define*

$$F \equiv F(a, b; c; x), \quad F_- \equiv F(a-1, b; c; x)$$

*on* $(0, 1)$. *Suppose that* $F^2 = \sum_{n=0}^{\infty} u_n x^n$ *is the Maclaurin series of* $F^2$, *where* $u_n = u_n(a, b, c)$ *depends on* $a, b, c$. *Then the coefficients* $u_n$ *satisfy the following recurrence relation*

$$u_{n+1} = \alpha_n u_n - \beta_n u_{n-1} \tag{2.3}$$

*for* $n \in \mathbb{N}$ *with* $u_0 = 1$ *and* $u_1 = 2ab/c$, *where*

$$\alpha_n = \frac{2n^3 + 3(a+b+c-1)n^2 + [(2a-1)(2b-1) + (a+b)(4c-1) - c]n + 2ab(2c-1)}{(n+1)(n+c)(n+2c-1)},$$
$$\beta_n = \frac{(n+2a-1)(n+2b-1)(n+a+b-1)}{(n+1)(n+c)(n+2c-1)}.$$

***Proof*** Let $FF_- = \sum_{n=0}^{\infty} v_n x^n$ and $F_-^2 = \sum_{n=0}^{\infty} w_n x^n$, where $v_n = v_n(a, b, c)$ and $w_n = w_n(a, b, c)$ are the Maclaurin coefficients of $FF_-$ and $F_-^2$, respectively. It is clear that $u_0 = v_0 = w_0 = 1$.

By [6, Theorem 3.12] (see also [15, p.86]), we have the following derivative formulas

$$\frac{dF}{dx} = \frac{(c-a)F_- + (a-c+bx)F}{x(1-x)} \quad \text{and} \quad \frac{dF_-}{dx} = \frac{(a-1)(F-F_-)}{x}. \tag{2.4}$$

By differentiation and (2.4),

$$\frac{d(F^2)}{dx} = \frac{2(c-a)FF_- + 2(a-c+bx)F^2}{x(1-x)} = \sum_{n=1}^{\infty} n u_n x^{n-1}, \tag{2.5}$$

$$\frac{d(FF_-)}{dx} = \frac{(c-a)F_-^2 + [1-c+(a+b-1)x]FF_- + (a-1)(1-x)F^2}{x(1-x)} = \sum_{n=1}^{\infty} n v_n x^{n-1}, \tag{2.6}$$

$$\frac{d(F_-^2)}{dx} = \frac{2(a-1)}{x}(FF_- - F_-^2) = \sum_{n=1}^{\infty} n w_n x^{n-1}. \tag{2.7}$$

Multiplying two sides of (2.5), (2.6) by $x(1-x)$ and two sides of (2.7) by $x$, we obtain

$$2(c-a)\sum_{n=0}^{\infty} v_n x^n + 2(a-c+bx)\sum_{n=0}^{\infty} u_n x^n$$
$$= 2(c-a)\sum_{n=1}^{\infty} v_n x^n + 2(a-c)\sum_{n=1}^{\infty} u_n x^n + 2b\sum_{n=0}^{\infty} u_n x^{n+1} \tag{2.8}$$

$$= \sum_{n=1}^{\infty} \big[2(c-a)v_n + 2(a-c)u_n + 2bu_{n-1}\big] x^n = \sum_{n=1}^{\infty} \big[nu_n - (n-1)u_{n-1}\big] x^n,$$

$$(c-a)\sum_{n=0}^{\infty} w_n x^n + [1-c+(a+b-1)x]\sum_{n=0}^{\infty} v_n x^n + (a-1)(1-x)\sum_{n=0} u_n x^n$$
$$= \sum_{n=1}^{\infty} \big[(c-a)w_n + (1-c)v_n + (a+b-1)v_{n-1} + (a-1)(u_n - u_{n-1})\big] x^n$$
$$= \sum_{n=1}^{\infty} \big[nv_n - (n-1)v_{n-1}\big] x^n \tag{2.9}$$

and

$$2(a-1)\sum_{n=1}^{\infty} [v_n - w_n] x^n = \sum_{n=1}^{\infty} nw_n x^n. \tag{2.10}$$

Comparing the coefficients of $x^n$ in (2.8), (2.9) and (2.10) leads to

$$2(c-a)v_n + 2(a-c)u_n + 2bu_{n-1} = nu_n - (n-1)u_{n-1}, \tag{2.11}$$
$$(c-a)w_n + (1-c)v_n + (a+b-1)v_{n-1} + (a-1)(u_n - u_{n-1}) = nv_n - (n-1)v_{n-1}, \tag{2.12}$$
$$2(a-1)(v_n - w_n) = nw_n. \tag{2.13}$$

By solving $w_n$ from (2.13) and substituting into (2.12), we obtain

$$\left[\frac{2(a-1)(c-a)}{n+2(a-1)} + 1 - c - n\right] v_n + (a+b+n-2)v_{n-1} = (1-a)(u_n - u_{n-1}),$$

or equivalently,

$$\left[\frac{2(a-1)(c-a)}{n+2a-1} - c - n\right] v_{n+1} + (a+b+n-1)v_n = (1-a)(u_{n+1} - u_n).$$

By (2.11), substituting the expressions of $v_n$ and $v_{n+1}$ into the above gives the recurrence relation of (2.3). □

We now state the following lemma as a special case of Proposition 2.3. In the rest of this article, we remind that the notations of $u_n$, $\alpha_n$ and $\beta_n$ are always expressed as in Lemma 2.4 if no risk for confusion.

**Lemma 2.4** *For $a \in (0, 1/2]$, suppose that $[F(a, 1-a; 2; x)]^2$ has the Maclaurin series expansion $\sum_{n=0}^{\infty} u_n x^n$, where $u_n = u_n(a, 1-a, 2)$ is defined in Proposition 2.3. Then the coefficients $u_n > 0$ for $n \in \mathbb{N}_0$ with $u_0 = 1$ and $u_1 = a(1-a)$, and satisfy the following recurrence relation*

$$u_{n+1} = \alpha_n u_n - \beta_n u_{n-1} \quad for\ n \in \mathbb{N}, \tag{2.14}$$

*where*

$$\alpha_n = \frac{2[n^3 + 3n^2 + 2(1+a-a^2)n + 3a(1-a)]}{(n+1)(n+2)(n+3)}, \quad \beta_n = \frac{n[n^2 - (1-2a)^2]}{(n+1)(n+2)(n+3)}.$$

***Proof*** From Proposition 2.3, it remains to show $u_n > 0$ for $n \geq \mathbb{N}_0$. Applying the Cauchy product of series,

$$[F(a, 1-a; 2; x)]^2 = \left[\sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(n!)^2} x^n\right]^2 = \sum_{n=0}^{\infty} u_n x^n,$$

where

$$u_n = \sum_{k=0}^{n} \frac{(a)_k(1-a)_k(a)_{n-k}(1-a)_{n-k}}{(k!)^2[(n-k)!]^2} > 0$$

for $a \in (0, 1/2]$. □

**Lemma 2.5** *Let $a \in (0, 1/2]$, $c \in (0, \infty)$ and define the sequence $\mathcal{Q}_n(a, c)$ for $n \in \mathbb{N}$ by*

$$\mathcal{Q}_n \equiv \mathcal{Q}_n(a, c) = -c^2n^3 + \left[3 + 2a - 2a^2)c^2 - 6\right]n^2 + \left[9a(1-a) - (2 + 3a - 3a^2)c^2\right]n + 2a(1-a)(1-a+a^2) + 2a(1-a)(1+a-a^2)c^2.$$

*Then we have the following conclusions:*

(1) $\mathcal{Q}_n(a, c) < 0$ *for* $n \geq 3$ *and* $(a, c) \in (0, 1/2] \times (0, \infty)$.
(2) *If* $(a, c) \in (0, 1/2] \times (0, c_1(a)]$, *then* $\mathcal{Q}_1(a, c) \leq 0$ *and* $\mathcal{Q}_2(a, c) < 0$; *if* $(a, c) \in (0, 1/2] \times [c_2(a), \infty)$, *then* $\mathcal{Q}_1(a, c) > 0$. *In particular,* $1 < c_1(a) \leq c_2(a)$ *for* $a \in (0, 1/2]$ *with the equality only for* $a = 1/2$, *where*

$$c_1(a) = \sqrt{\frac{6 - 11a + 13a^2 - 4a^3 + 2a^4}{a(1-a)(1+2a-2a^2)}}, \quad c_2(a) = \frac{1}{a(1-a)} - 1.$$

***Proof*** (1) Let

$$\Delta\mathcal{Q}_n = \mathcal{Q}_{n+1} - \mathcal{Q}_n = -3c^2n^2 + \left[(3 + 4a - 4a^2)c^2 - 12\right]n - a(1-a)c^2 - 3(2 - 3a + 3a^2),$$
$$\Delta^2\mathcal{Q}_n = \Delta\mathcal{Q}_{n+1} - \Delta\mathcal{Q}_n = -6c^2n + 4a(1-a)c^2 - 12.$$

For $(a, c) \in (0, 1/2] \times (0, \infty)$ and $n \geq 1$, it can be easily seen that

$$\Delta^2\mathcal{Q}_n \leq -6c^2 + 4a(1-a)c^2 - 12 = -\left[(6 - 4a + 4a^2)c^2 + 12\right] < 0.$$

In other words, $\Delta\mathcal{Q}_n$ is strictly decreasing for $n \geq 1$. This gives

$$\Delta\mathcal{Q}_n \leq \Delta\mathcal{Q}_2 = -\left[30 - 9a + 9a^2 + \frac{7(1-2a)^2 + 17}{4}c^2\right] < 0$$

and so $\mathcal{Q}_n$ is strictly decreasing for $n \geq 2$. Similarly, we obtain

$$\mathcal{Q}_n(a, c) \leq \mathcal{Q}_3 = -\left[\frac{(1-2a)^2(57 - 4a + 4a^2) + 375}{8} + (3-a)(2+a)(1 - 2a + 2a^2)c^2\right] < 0$$

for $n \geq 3$. This completes the proof of (1).
(2) If $(a, c) \in (0, 1/2] \times (0, c_1(a)]$, then we can directly verify that

$$\mathcal{Q}_1(a, c) = -6 + 11a - 13a^2 + 4a^3 - 2a^4 + a(1-a)(1+2a-2a^2)c^2 \leq \mathcal{Q}_1(a, c_1(a)) = 0$$

and

$$\begin{aligned}\mathcal{Q}_2(a,c) &= -24+20a-22a^2+4a^3-2a^4+2a(1-a^2)(2-a)c^2\\ &\le \mathcal{Q}_2(a,c_1(a)) = -\frac{12a(1-a)(5-2a+2a^2)}{1+2a-2a^2} < 0.\end{aligned}$$

If $(a,c)\in(0,1/2]\times[c_2(a),\infty)$, then we have

$$\begin{aligned}\mathcal{Q}_1(a,c) &= -6+11a-13a^2+4a^3-2a^4+a(1-a)(1+2a-2a^2)c^2\\ &\ge \mathcal{Q}_1(a,c_2(a)) = \frac{(1-2a)^2(1-2a+2a^2)}{a(1-a)} > 0.\end{aligned}$$

Finally, inequalities $c_2(a)\ge c_1(a)>1$ for $a\in(0,1/2]$ follows easily from

$$c_2^2(a)-c_1^2(a) = \frac{(1-2a)^2(1-2a+2a^2)}{a^2(1-a)^2(1+2a-2a^2)} \ge 0, \quad c_1^2(a)-1 = \frac{6(1-2a+2a^2)}{a(1-a)(1+2a-2a^2)} > 0.$$

□

**Lemma 2.6** *For* $a\in(0,1/2]$ *and* $c\in(0,\infty)$*, the function*

$$x\mapsto \eta(x)\equiv\eta(x,a,c) = \frac{R(a)-\log(1-x)}{\log(1+c/\sqrt{1-x})}$$

*is strictly decreasing on* $(0,1)$ *if and only if* $\varrho(c)\le R(a)$*, where* $\varrho(c)=2(1+1/c)\log(1+c)$*. In particular, if* $0<c\le 1$*, then* $\varrho(c)\le R(a)$ *for* $a\in(0,1/2]$*.*

***Proof*** Differentiation yields

$$\eta'(x) = \frac{\hat{\eta}(x)}{2(1-x)(c+\sqrt{1-x})\left[\log(1+c/\sqrt{1-x})\right]^2}, \tag{2.15}$$

where

$$\hat{\eta}(x) = 2(c+\sqrt{1-x})\log\left(1+\frac{c}{\sqrt{1-x}}\right) - c\left[R(a)-\log(1-x)\right].$$

Let $\varrho(c)=2(1+1/c)\log(1+c)$. It is clear that $\hat{\eta}(0)=c\left[\varrho(c)-R(a)\right]$. Moreover,

$$\hat{\eta}'(x) = -\frac{\log(1+c/\sqrt{1-x})}{\sqrt{1-x}} < 0$$

for $x\in(0,1)$ and $\hat{\eta}(x)$ is strictly decreasing on $(0,1)$. By (2.15), $\eta(x)$ is strictly decreasing on $(0,1)$ if and only if $\hat{\eta}(0)\le 0$, namely, $\varrho(c)\le R(a)$.

In particular, it can be easily verified that the function

$$c\mapsto \frac{2(1+c)\log(1+c)}{c}$$

is strictly increasing on $(0,\infty)$ by L'Hôpital Monotone Rule. This together with (1.6) gives

$$\varrho(c)\le\varrho(1)=4\log 2=R(1/2)\le R(a) \quad \text{for } 0<c\le 1,$$

since $R(a)$ is strictly decreasing on $(0,1/2]$ (see [48, Theorem 1(1)] and also [50, Lemma 2.1]). □

**Lemma 2.7** *For* $a \in (0, 1/2]$, *the function*

$$g(x) = \frac{2x(1-x)\mathcal{K}_a(\sqrt{x}) - (1-a)(1-2x)\left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right]}{x\left[(1-a)\mathcal{E}_a(\sqrt{x}) + a(1-x)\mathcal{K}_a(\sqrt{x}))\right]}$$

*is strictly decreasing from* $(0, 1)$ *onto* $(1, a^2 - a + 2)$.

***Proof*** Let

$$\begin{aligned} g_1(x) &= 2x(1-x)\mathcal{K}_a(\sqrt{x}) - (1-a)(1-2x)\left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right], \\ g_2(x) &= x\left[(1-a)\mathcal{E}_a(\sqrt{x}) + a(1-x)\mathcal{K}_a(\sqrt{x}))\right]. \end{aligned}$$

By (1.4) and (1.5), we can rewrite $g(x)$, in terms of power series, as

$$g(x) = \frac{g_1(x)}{g_2(x)} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n},$$

where

$$\begin{aligned} a_0 &= a^2 - a + 2, \quad a_n = -\frac{(a)_{n-1}(1-a)_{n-1}}{n!(n+1)!} p_n \quad (n \geq 1), \\ b_0 &= 1, \quad b_n = -\frac{(a)_{n-1}(1-a)_{n-1}}{n!(n+1)!} q_n \quad (n \geq 1) \end{aligned}$$

and

$$\begin{aligned} p_n &= (a^2 - a + 2)n^2 + (5a^2 - 5a + 2)n - a(1-a)(2 - a + a^2), \\ q_n &= (n+1)[(a^2 - a + 1)n - a(1-a)]. \end{aligned}$$

For $a \in (0, 1/2]$ and $n \geq 1$, it can be easily verified that

$$\begin{aligned} p_n &\geq [(a^2 - a + 2) + (5a^2 - 5a + 2)]n - a(1-a)(2 - a + a^2) \\ &\geq 2(2 - 3a + 3a^2) - a(1-a)(2 - a + a^2) = \frac{33 + (1-2a)^2(31 - 4a + 4a^2)}{16} > 0 \end{aligned}$$

and

$$q_n \geq (n+1)[(a^2 - a + 1) - a(1-a)] = (n+1)(1 - 2a + 2a^2) > 0.$$

Since $a_n/b_n = p_n/q_n$ and $p_n, q_n > 0$ for $n \geq 1$, it follows that

$$\operatorname{sgn}\left(\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n}\right) = \operatorname{sgn}(p_{n+1}q_n - p_n q_{n+1}). \tag{2.16}$$

Moreover,

$$\begin{aligned} &\frac{p_{n+1}q_n - p_n q_{n+1}}{a(1-a)} \\ &= (2 - 3a + 3a^2)n^2 + \frac{3 + (1-2a)^2(13 - 4a + 4a^2)}{8}n - a(1-a)(2 - 3a + 3a^2) \\ &\geq (2 - 3a + 3a^2) + \frac{3 + (1-2a)^2(13 - 4a + 4a^2)}{8} - a(1-a)(2 - 3a + 3a^2) \\ &= \frac{21 + (1-2a)^2(43 - 20a + 20a^2)}{16} > 0. \end{aligned}$$

This in conjunction with (2.16) implies $a_n/b_n$ is strictly increasing for $n \in \mathbb{N}$.

Since $b_n < 0$ for $n \in \mathbb{N}$, it follows from (2.1) that

$$\operatorname{sgn}(g_1/g_2)' = \operatorname{sgn}(g_2') \cdot \operatorname{sgn}(H_{g_1,g_2}) = -\operatorname{sgn}(H_{g_1,g_2}). \tag{2.17}$$

According to this with (2.2) and $g_2(x) > 0$, it suffices to show the monotonicity of $g_1'(x)/g_2'(x)$.

Making use of power series,

$$\frac{g_1'(x)}{g_2'(x)} = \frac{\sum_{n=1}^{\infty} n a_n x^{n-1}}{\sum_{n=1}^{\infty} n b_n x^{n-1}} = \frac{\sum_{n=0}^{\infty} a_n' x^n}{\sum_{n=0}^{\infty} b_n' x^n},$$

where $a_n' = -(n+1)a_{n+1}$ and $b_n' = -(n+1)b_{n+1}$. Since $a_n', b_n' > 0$ and $a_n'/b_n' = a_{n+1}/b_{n+1}$, Lemma 2.1(1) and the monotonicity of $\{a_n/b_n\}_{n=1}^{\infty}$ lead to the conclusion that $g_1'(x)/g_2'(x)$ is strictly increasing on $(0, 1)$ and so is $H_{g_1,g_2}(x)$. This gives

$$H_{g_1,g_2}(x) > H_{g_1,g_2}(0^+) = \frac{a_1}{b_1} b_0 - a_0 = \frac{a(1-a)(3a^2 - 3a + 2)}{2(2a^2 - 2a + 1)} > 0.$$

Combining this with (2.17) implies the monotonicity of $g(x)$. Two limiting values are clear. □

**Lemma 2.8** *For $a \in (0, 1/2]$, the function*

$$\varphi(r) = \frac{a r^2 r'^2 \mathcal{K}_a^2 - (1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)^2 - (r'^2 - r^2)\mathcal{K}_a(\mathcal{E}_a - r'^2\mathcal{K}_a)}{r^4 \mathcal{K}_a^2}$$

*is strictly decreasing from $(0, 1)$ onto $(0, a(a^2 - a + 2)/2)$.*

***Proof*** Let $\varphi_1(r) = a r^2 r'^2 \mathcal{K}_a^2 - (1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)^2 - (r'^2 - r^2)\mathcal{K}_a(\mathcal{E}_a - r'^2\mathcal{K}_a)$ and $\varphi_2(r) = r^4\mathcal{K}_a^2$. Then we clearly see that $\varphi(r) = \varphi_1(r)/\varphi_2(r)$ and $\varphi_1(0) = \varphi_2(0) = 0$.

Differentiation leads to

$$\begin{aligned}
\varphi_1'(r) &= 4r\mathcal{K}_a^2(\mathcal{E}_a - r'^2\mathcal{K}_a) - 2(1-a)(r'^2 - r^2)\frac{(\mathcal{E}_a - r'^2\mathcal{K}_a)^2}{rr'^2} \\
&= \frac{2(\mathcal{E}_a - r'^2\mathcal{K}_a)}{rr'^2}\Big[2r^2r'^2\mathcal{K}_a - (1-a)(r'^2 - r^2)(\mathcal{E}_a - r'^2\mathcal{K}_a)\Big], \\
\varphi_2'(r) &= 4r^3\mathcal{K}_a^2 + 2r^3\mathcal{K}_a\frac{2(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{r'^2} = \frac{4r^3\mathcal{K}_a}{r'^2}\Big[(1-a)\mathcal{E}_a + ar'^2\mathcal{K}_a\Big].
\end{aligned}$$

By simplifying, we obtain

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{\mathcal{E}_a - r'^2\mathcal{K}_a}{2r^2\mathcal{K}_a} g(r^2), \tag{2.18}$$

where $g(x)$ is defined in Lemma 2.7.

Lemmas 2.2(1), 2.7 and (2.18) lead to the conclusion that $\varphi_1'(r)/\varphi_2'(r)$ is strictly decreasing on $(0, 1)$, and so is $\varphi(r)$ by l'Hôpital Monotone Rule (see [4, Theorem 1.25]). It is clear that $\varphi(1^-) = 0$. By L'Hôpital's Rule, it follows from Lemma 2.2(1) and 2.7 that

$$\varphi(0^+) = \lim_{r \to 0^+} \frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{a(a^2 - a + 2)}{2}.$$

□

**Lemma 2.9** *Let* $a \in (0, 1/2]$ *and define the function* $\Phi(r)$ *on* $(0, 1)$ *by*

$$\Phi(r) \equiv \Phi(r, a, \lambda) = \frac{(1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)}{r^2 r'^2 \mathcal{K}_a} - \frac{\lambda}{r'^2}.$$

*Then* $\Phi$ *is strictly decreasing if and only if* $\lambda \geq [a(1-a)(a^2-a+2)]/2$ *and strictly increasing if and only if* $\lambda \leq 0$. *If* $0 < \lambda < [a(1-a)(a^2-a+2)]/2$, *then* $\Phi$ *is piecewise monotone on* $(0, 1)$.

***Proof*** Differentiating $\Phi(r)$ yields

$$\begin{aligned}\Phi'(r) &= (1-a)\frac{2ar^3r'^2\mathcal{K}_a^2 - (\mathcal{E}_a - r'^2\mathcal{K}_a)[2rr'^2\mathcal{K}_a - 2r^3\mathcal{K}_a + 2(1-a)r(\mathcal{E}_a - r'^2\mathcal{K}_a)]}{r^4r'^4\mathcal{K}_a^2} - \frac{2\lambda r}{r'^4}\\ &= \frac{2(1-a)r}{r'^4}\left[\frac{ar^2r'^2\mathcal{K}_a^2 - (1-a)(\mathcal{E}_a - r'^2\mathcal{K}_a)^2 - (r'^2 - r^2)\mathcal{K}_a(\mathcal{E}_a - r'^2\mathcal{K}_a)}{r^4\mathcal{K}_a^2} - \frac{\lambda}{1-a}\right].\end{aligned}$$

This together with Lemma 2.8 completes the proof of Lemma 2.9. □

**Lemma 2.10** *Let* $a \in (0, 1/2]$ *and define* $h(x)$ *on* $(0, 1)$ *by*

$$h(x) = \frac{\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x}) - (1-x)\left[(1+(2a-1)x)\mathcal{K}_a(\sqrt{x}) - \mathcal{E}_a(\sqrt{x})\right]}{\left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right]^2}.$$

*Then the following statements are true:*

(1) $h(x)$ *is strictly increasing on* $(0, 1)$ *if and only if* $a = 1/2$*;*
(2) *If* $a \in (0, 1/2)$, *then there exists a number* $x_1 \in (0, 1)$ *such that* $h(x)$ *is strictly decreasing on* $(0, x_1)$ *and strictly increasing on* $(x_1, 1)$.

*Moreover,* $h(0^+) = 2(1-a+a^2)/(\pi a)$ *and* $h(1^-) = 2(1-a)/\sin(\pi a)$ *with* $h(0^+) < h(1^-)$.

***Proof*** Let $h_1(x) = \mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x}) - (1-x)\left[(1+(2a-1)x)\mathcal{K}_a(\sqrt{x}) - \mathcal{E}_a(\sqrt{x})\right]$ and $h_2(x) = \left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right]^2$.

By (1.4) and (1.5), one has

$$h_1(x) = \frac{\pi}{2}ax^2\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n(2-2a+2a^2+3n)}{n!(n+2)!}x^n, \quad h_2(x) = \frac{\pi^2}{4}a^2x^2\left[F(a, 1-a; 2; x)\right]^2.$$

Applying Lemma 2.4, we rewrite $h(x)$ as

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{2}{\pi a}\frac{\sum_{n=0}^{\infty}v_nx^n}{\sum_{n=0}^{\infty}u_nx^n},$$

where $u_n$ is defined as Lemma 2.4 and

$$v_n = \frac{(a)_n(1-a)_n(2-2a+2a^2+3n)}{n!(n+2)!}.$$

Since $v_n > 0$ and $u_n > 0$, it can be easily seen that

$$\operatorname{sgn}\left(\frac{v_{n+1}}{u_{n+1}} - \frac{v_n}{u_n}\right) = -\operatorname{sgn}\left(u_{n+1} - \frac{v_{n+1}}{v_n}u_n\right). \tag{2.19}$$

By (2.14), we denote by

$$d_n = u_{n+1} - \frac{v_{n+1}}{v_n}u_n = \hat{\alpha}_nu_n - \beta_nu_{n-1}, \tag{2.20}$$

where

$$\hat{\alpha}_n = \alpha_n - \frac{v_{n+1}}{v_n} = \frac{\kappa_n(a)}{(n+1)(n+2)(n+3)(3n+2a^2-2a+2)}$$

and $\kappa_n(a) = 3n^4 + 2(a^2-a+4)n^3 + 3(1+a-a^2)n^2 - (6a^4 - 12a^3 + 17a^2 - 11a + 2)n + 2a(1-a)(1-2a)^2$.

Due to

$$\begin{aligned}\kappa_{n+1}(a) - \kappa_n(a) &= 12n^3 + 6(a^2-a+7)n^2 + 42n + 6[2 + a(1-a)(2-a+a^2)]\\ &\geq 12 + 6(a^2-a+7) + 42 + 6[2 + a(1-a)(2-a+a^2)]\\ &= 108 + 6a(1-a)(1-a+a^2) > 0\end{aligned}$$

for $a \in (0, 1/2]$ and $n \geq \mathbb{N}$, it follows that

$$\kappa_n(a) \geq \kappa_1(a) = 2[6 + 7a(1-a)(1+a-a^2)] > 0$$

for $n \in \mathbb{N}$ and so $\hat{\alpha}_n > 0$ for $n \in \mathbb{N}$.

For $a \in (0, 1/2]$, we now prove $d_n < 0$ for $n \in \mathbb{N}$ by mathematical induction.

By (2.20) and Lemma 2.4, we can verify

$$d_1 = -\frac{a(1-a)(1-a+a^2)(4-7a+7a^2)}{12(5-2a+2a^2)} < 0.$$

Assume that $d_n < 0$ for $n \geq 1$, that is

$$u_n < \frac{\beta_n}{\hat{\alpha}_n} u_{n-1}, \tag{2.21}$$

then it follows from (2.14) and (2.20) together with $\hat{\alpha}_n > 0$, $\beta_n > 0$ and $u_n > 0$ that

$$\begin{aligned}d_{n+1} &= \hat{\alpha}_{n+1}u_{n+1} - \beta_{n+1}u_n = \hat{\alpha}_{n+1}(\alpha_n u_n - \beta_n u_{n-1}) - \beta_{n+1}u_n\\ &= (\hat{\alpha}_{n+1}\alpha_n - \beta_{n+1})u_n - \hat{\alpha}_{n+1}\beta_n u_{n-1} < 0,\end{aligned}$$

if $\hat{\alpha}_{n+1}\alpha_n - \beta_{n+1} \leq 0$. Otherwise, if $\hat{\alpha}_{n+1}\alpha_n - \beta_{n+1} > 0$, combining this with (2.21), we obtain

$$d_{n+1} < (\hat{\alpha}_{n+1}\alpha_n - \beta_{n+1})\frac{\beta_n}{\hat{\alpha}_n}u_{n-1} - \hat{\alpha}_{n+1}\beta_n u_{n-1} = \frac{\beta_n}{\hat{\alpha}_n}\left(\hat{\alpha}_{n+1}\frac{v_{n+1}}{v_n} - \beta_{n+1}\right)u_{n-1} < 0,$$

since

$$\hat{\alpha}_{n+1}\frac{v_{n+1}}{v_n} - \beta_{n+1} = -\frac{a(1-a)\hat{\kappa}_n(a)}{(n+1)(n+2)(n+3)^2(n+3)(3n+2-2a+2a^2)}$$

and

$$\begin{aligned}\hat{\kappa}_n(a) &= 6n^3 + (20 - 17a + 17a^2)n^2 + \frac{1}{8}\Big[105 + (1-2a)^2(103 - 12a + 12a^2)\Big]n\\ &\quad + 2(6 - 19a + 26a^2 - 14a^3 + 7a^4)\\ &\geq \hat{\kappa}_1(a) = \frac{1}{4}\Big[153 + (1-2a)^2(103 - 20a + 20a^2)\Big] > 0.\end{aligned}$$

By mathematical induction, we show that $d_n < 0$ for $n \in \mathbb{N}$ and so the sequence $\{v_n/u_n\}$ is strictly increasing for $n \in \mathbb{N}$ by (2.19) and (2.20).

On the other hand, by Lemma 2.4, it is easy to obtain

$$\frac{v_1}{u_1}-\frac{v_0}{u_0}=-\frac{(1-2a)^2}{6}\leq 0 \tag{2.22}$$

with the equal only for $a=1/2$.

(1) If $a=1/2$, then it follows from (2.22) that $\{v_n/u_n\}$ is strictly increasing for $n\in\mathbb{N}_0$. This together with Lemma 2.1(1) shows $h(x)$ is strictly increasing on $(0,1)$. Conversely, the monotonicity of $h(x)$ requires us to satisfy

$$h'(0^+)=\lim_{x\to 0^+}\frac{2}{\pi a}\left(\frac{\sum_{n=0}^{\infty}v_nx^n}{\sum_{n=0}^{\infty}u_nx^n}\right)'=\frac{2}{\pi a}\frac{v_1u_0-v_0u_1}{u_0^2}=-\frac{(1-a)(1-2a)^2}{3\pi}\geq 0,$$

that is $a=1/2$.

(2) If $a\in(0,1/2)$, then it follows from (2.22) that $\{v_n/u_n\}$ is strictly decreasing for $0\leq n\leq 1$ and $\{v_n/u_n\}$ is strictly increasing for $n\geq 1$. Moreover, by Lemma 2.2(1),

$$\begin{aligned}H_{h_1,h_2}(1^-)&=\lim_{x\to 1^-}\left\{\frac{3a\mathcal{K}_a(\sqrt{x})-(1+2a-2a^2)\left[\mathcal{E}_a(\sqrt{x})-(1-x)\mathcal{K}_a(\sqrt{x})\right]}{2a\mathcal{K}_a(\sqrt{x})\left[\mathcal{E}_a(\sqrt{x})-(1-x)\mathcal{K}_a(\sqrt{x})\right]}h_2(x)-h_1(x)\right\}\\&=\frac{\sin(\pi a)}{4(1-a)}>0.\end{aligned}$$

This in conjunction with Lemma 2.1(2) shows that there exists a number $x_1\in(0,1)$ such that $h(x)$ is strictly decreasing on $(0,x_1)$ and strictly increasing on $(x_1,1)$.
It is clear for $h(0^+)$ and $h(1^-)$. Moreover, by [49, 4.3.68],

$$\begin{aligned}\frac{\pi a}{2(1-a)}\left[h(1^-)-h(0^-)\right]&=\frac{\pi a}{\sin(\pi a)}-\frac{1-a+a^2}{1-a}>1+\frac{\pi^2a^2}{6}+\frac{7\pi^4a^4}{360}-\frac{1-a+a^2}{1-a}\\&>\frac{a^2\left[1+(1-2a)(23-18a+36a^2)\right]}{40(1-a)}>0.\end{aligned}$$

□

**Lemma 2.11** *Let $a\in(0,1/2]$, $c\in(0,\infty)$ and define $\zeta(r)$ on $(0,1)$ by*

$$\zeta(r)\equiv\zeta(r,a,c)=\frac{(c+r')\log(1+c/r')-cr^2\mathcal{K}_a/\left[2(1-a)(\mathcal{E}_a-r'^2\mathcal{K}_a)\right]}{r'}.$$

*is strictly increasing and positive on $(0,1)$ if $c\geq\max_{a\in(0,1/2]}\{c^*(a),e^{R(a)/2}\}$, where $c^*(a)$ is defined as in Theorem 1.2.*

***Proof*** Differentiation yields

$$\frac{d[(c+r')\log(1+c/r')]}{dr}=\frac{r}{r'^2}\left[c-r'\log(1+c/r')\right],$$

$$\begin{aligned}\frac{d}{dr}\left[\frac{cr^2\mathcal{K}_a}{2(1-a)(\mathcal{E}_a-r'^2\mathcal{K}_a)}\right]&=\frac{c}{2(1-a)}\frac{\left[2r\mathcal{K}_a+r^2\cdot\frac{2(1-a)(\mathcal{E}_a-r'^2\mathcal{K}_a)}{rr'^2}\right](\mathcal{E}_a-r'^2\mathcal{K}_a)-r^2\mathcal{K}_a\cdot 2ar\mathcal{K}_a}{(\mathcal{E}_a-r'^2\mathcal{K}_a)^2}\\&=\frac{c}{2(1-a)}\frac{r\left[(2(1-a)\mathcal{E}_a+2ar'^2\mathcal{K}_a)(\mathcal{E}_a-r'^2\mathcal{K}_a)-2ar^2r'^2\mathcal{K}_a^2\right]}{r'^2(\mathcal{E}_a-r'^2\mathcal{K}_a)^2}\end{aligned}$$

and thereby

$$
\begin{aligned}
&r'^2\zeta'(r)\\
&=\frac{r}{r'}\Big[c-r'\log(1+c/r')\Big]-\frac{cr\left[(2(1-a)\mathcal{E}_a+2ar'^2\mathcal{K}_a)(\mathcal{E}_a-r'^2\mathcal{K}_a)-2ar^2r'^2\mathcal{K}_a^2\right]}{2(1-a)r'(\mathcal{E}_a-r'^2\mathcal{K}_a)^2}\\
&+\frac{r}{r'}\left[(c+r')\log(1+c/r')-\frac{cr^2\mathcal{K}_a}{2(1-a)(\mathcal{E}_a-r'^2\mathcal{K}_a)}\right]\\
&=\frac{cr}{r'(\mathcal{E}_a-r'^2\mathcal{K}_a)^2}\left\{(\mathcal{E}_a-r'^2\mathcal{K}_a)^2\log(1+c/r')-\frac{\mathcal{K}_a\left[\mathcal{E}_a-r'^2\mathcal{K}_a-r'^2\left((r'^2+2ar^2)\mathcal{K}_a-\mathcal{E}_a\right)\right]}{2(1-a)}\right\}\\
&=\frac{cr\mathcal{K}_a}{r'}\left[\frac{1}{F(r^2)}-\frac{h(r^2)}{2(1-a)}\right],
\end{aligned}
\tag{2.23}
$$

where $F(x)$ and $h(x)$ are defined as in Theorem 1.2 and Lemma 2.10, respectively.

If $c\ge\max_{a\in(0,1/2]}\left\{c^*(a),e^{R(a)/2}\right\}$, then by Theorem 1.2(1) and Lemma 2.10 we obtain

$$
\frac{1}{F(r^2)}-\frac{h(r^2)}{2(1-a)}>\frac{1}{F(1^-)}-\frac{1}{2(1-a)}\max\{h(0^+),h(1^-)\}=0.
$$

This together with (2.23) gives the monotonicity of $\zeta(r)$. Moreover, by Lemma 2.2(1),

$$
\zeta(0^+)=(1+c)\log(1+c)-\frac{c}{2a(1-a)}=\frac{c}{2}\rho^*(c;a)\ge 0,
$$

since it has been shown that $\rho^*(c;a)\ge 0$ for $c\ge c^*(a)$ in the proof of Theorem 1.2. This completes the proof. □

## 3 Proofs of Theorems 1.2, 1.3 and 1.4

***Proof of Theorems 1.2*** Let us denote $f_1(x)=F(a,1-a;1;x)$ and $f_2(x)=\log(1+c/\sqrt{1-x})$ for $a\in(0,1/2]$. Then it suffices to show the monotonicity of $f_1(x)/f_2(x)$ on $(0,1)$.

By differentiation and (1.1), (1.2), we obtain

$$
f_1'(x)=\frac{a(1-a)F(a,1-a;2;x)}{1-x}\quad\text{and}\quad f_2'(x)=\frac{c(c-\sqrt{1-x})}{2(c^2-1+x)(1-x)},
$$

which yields

$$
\frac{f_1'(x)}{f_2'(x)}=\frac{2a(1-a)(c^2-1+x)F(a,1-a;2;x)}{c(c-\sqrt{1-x})}\triangleq\frac{2a(1-a)}{c}\frac{\hat{f}_1(x)}{\hat{f}_2(x)}.
$$

Moreover, we can rewrite $\hat{f}_1(x)/\hat{f}_2(x)$, in terms of power series, as

$$
\frac{\hat{f}_1(x)}{\hat{f}_2(x)}=\frac{(c^2-1+x)\sum_{n=0}^{\infty}\frac{(a)_n(1-a)_n}{n!(n+1)!}x^n}{c-\sum_{n=0}^{\infty}\frac{(-1/2)_n}{n!}x^n}=\frac{\sum_{n=0}^{\infty}A_nx^n}{\sum_{n=0}^{\infty}B_nx^n},
$$

where

$$
A_0=c^2-1,\quad A_n=\frac{(a)_{n-1}(1-a)_{n-1}}{n!(n+1)!}\Big[c^2n^2+(2-c^2)n+a(1-a)(c^2-1)\Big]\quad(n\ge 1)
$$

and

$$B_0 = c - 1, \quad b_n = \frac{(1/2)_{n-1}}{2n!} \quad (n \geq 1).$$

By Lemma 2.1, we need to study the monotonicity of the sequence $\{C_n = A_n/B_n\}_{n=0}^{\infty}$ to show the monotonicity of $\hat{f}_1(x)/\hat{f}_2(x)$.

Elementary calculations lead to

$$C_1 - C_0 = (c-1)\big[a(1-a)c + a(1-a) - 1\big] \tag{3.1}$$

and

$$C_{n+1} - C_n = \frac{(a)_{n-1}(1-a)_{n-1}}{(1/2)_n(n+2)!} \mathcal{Q}_n(a,c) \qquad (n \geq 1). \tag{3.2}$$

Let $\rho_*(a,c) = \varrho(c) - R(a)$ and $\rho^*(a,c) = \varrho(c) - 1/[a(1-a)]$, where $\varrho(c)$ is defined as in Lemma 2.6. As in proof of Lemma 2.6, it can be easily seen that $\rho_*(a,c)$ and $\rho^*(a,c)$ are strictly increasing for $c \in (0,\infty)$. Clearly, $\rho_*(a,0^+) = 2 - R(a) \leq 2 - 4\log 2 \approx -0.7725$, $\rho^*(a,0^+) = 2 - 1/[a(1-a)] \leq -2$ for $a \in (0,1/2]$ and $\lim_{c\to\infty} \rho_*(a,c) = \lim_{c\to\infty} \rho^*(a,c) = \infty$. So there exist unique two numbers $c_*(a), c^*(a) \in (0,\infty)$ such that $\rho_*(a,c_*(a)) = 0$ and $\rho^*(a,c^*(a)) = 0$. Moreover, $\rho_*(a,c) < 0$ for $c \in (0,c_*(a))$ and $\rho_*(a,c) > 0$ for $c \in (c_*(a),\infty)$; $\rho^*(a,c) < 0$ for $c \in (0,c^*(a))$ and $\rho^*(a,c) > 0$ for $c \in (c^*(a),\infty)$.

It was proved in [50, Lemma 2.1] that $a \mapsto 1/[a(1-a)] - R(a)$ is strictly increasing from $(0,1/2]$ onto $(1, 4-4\log 2]$ and so $1/[a(1-a)] > R(a)$. Combining this with Lemma 2.6, we obtain $\rho_*(a,1) = \varrho(1) - R(a) \leq 0$ and $\rho_*(a,c^*(a)) = 1/[a(1-a)] - R(a) > 0$, that is to say,

$$1 \leq c_*(a) < c^*(a). \tag{3.3}$$

Let $\tilde{\rho}(a) = (1-a+a^2)\rho^*(a,c_2(a)) = 1 - 1/[a(1-a)] - 2\log[a(1-a)]$, where $c_2(a)$ is given as in Lemma 2.5. Then it follows from

$$\tilde{\rho}(1/2) = 2\log 4 - 3 \approx -0.2274 \quad \text{and} \quad \tilde{\rho}'(a) = \frac{(1-2a)(1-2a+2a^2)}{a^2(1-a)^2} > 0$$

that $\rho^*(a,c_2(a)) < 0$ and so by the property of $\rho^*(a,c)$ shows

$$c_2(a) < c^*(a). \tag{3.4}$$

(1) **Sufficiency**. If $c \geq \max\{c^*(a), e^{R(a)/2}\}$, then we have $c > c_2(a)$ by (3.4) and $\rho^*(a,c) \geq 0$, $\log c \geq R(a)/2$.
By (3.1) and (3.2), it follows from $c > c_2(a)$ and Lemma 2.5 that $C_1 > C_0$, $C_2 > C_1$ and $C_{n+1} < C_n$ for $n \geq 3$. In either case $C_3 \geq C_2$ or $C_3 < C_2$, there exists a number $n_0 = 2$ or 3 such that $C_n$ is increasing for $0 \leq n \leq n_0$ and $C_n$ is decreasing for $n \geq n_0$. Moreover, from (1.1) and (1.2) we clearly see that

$$\begin{aligned} &H_{\hat{f}_1,\hat{f}_2}(x) \\ &= 2\left[c\sqrt{1-x} - (1-x)\right]\left[\frac{1}{2}a(1-a)(c^2-1+x)F(1+a,2-a;3;x) + F(a,1-a;2;x)\right] \\ &\quad - (c^2-1+x)F(a,1-a;2;x) \to -\frac{c^2\sin(a\pi)}{a(1-a)\pi} < 0 \end{aligned} \tag{3.5}$$

as $x \to 1^-$.

Lemma 2.1(2) together with (3.5) and the piecewise monotonicity of $C_n$ implies that there exists $\delta_1 \in (0,1)$ such that $\hat{f}_1(x)/\hat{f}_2(x)$ is strictly increasing on $(0,\delta_1)$ and strictly decreasing on $(\delta_1,1)$, so is $f_1'(x)/f_2'(x)$. Due to (2.2) and $f_2(x)>0$, $H_{f_1,f_2}(x)$ is strictly increasing on $(0,\delta_1)$ and strictly decreasing on $(\delta_1,1)$.
Further, it can be obtained from (1.1) and (1.2) that

$$\begin{aligned} &H_{f_1,f_2}(0^+)\\ &=\lim_{x\to0^+}\left[\frac{2a(1-a)}{c}(c+\sqrt{1-x})F(a,1-a;2;x)\log\left(1+\frac{c}{\sqrt{1-x}}\right)-F(a,1-a;1;x)\right]\\ &=\frac{2a(1-a)(1+c)\log(1+c)-c}{c}=a(1-a)\rho^*(a,c) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} H_{f_1,f_2}(1^-)&=\lim_{x\to1^-}\left[\frac{2\sin(\pi a)}{\pi}\log\left(1+\frac{c}{\sqrt{1-x}}\right)-F(a,1-a;1;x)\right]\\ &=\lim_{x\to1^-}\frac{\sin(\pi a)}{\pi}\left[2\log\left(1+\frac{c}{\sqrt{1-x}}\right)+\log(1-x)-R(a)\right]\\ &=\frac{2\sin(\pi a)}{\pi}\left[\log c-\frac{R(a)}{2}\right]. \end{aligned} \tag{3.7}$$

Since $\rho^*(a,c)\ge 0$ and $\log c\ge R(a)/2$, we clearly see from (3.6) and (3.7) that $H_{f_1,f_2}(0^+)\ge 0$ and $H_{f_1,f_2}(1^-)\ge 0$. According to this with the piecewise monotonicity of $H_{f_1,f_2}(x)$, it can be easily seen that $H_{f_1,f_2}(x)>0$ for $x\in(0,1)$. This together with (2.1) and $f_2'(x)>0$ shows that $f_1(x)/f_2(x)$ is strictly increasing on $(0,1)$, so is $F(x)$.
**Necessity**. Due to $f_2'(x)>0$ and (2.1), the necessary condition for the monotonicity of $f_1(x)/f_2(x)$ requires us to satisfy

$$H_{f_1,f_2}(0^+)\ge 0 \quad\text{and}\quad H_{f_1,f_2}(1^-)\ge 0.$$

This in conjunction with (3.6) and (3.7) implies $\rho^*(a,c)\ge 0$ and $\log c\ge R(a)/2$. By the property of $\rho^*(a,c)$, we obtain $c\ge c^*(a)$ and $c\ge e^{R(a)/2}$, equivalently $c\ge \max_{a\in(0,1/2]}\{c^*(a),e^{R(a)/2}\}$.

(2) For $0<c\le\max_{a\in(0,1/2]}\{c_*(a),c_1(a)\}$, that is $0<c\le c_*(a)$ or $0<c\le c_1(a)$, equivalently, $0<c\le c_*(a)$ or $1<c\le c_1(a)$ by (3.3) and Lemma 2.5. We divide the proof into two cases.

Case 1: $0<c\le c_*(a)$. In this case, it follows from the property of $\rho_*(a,c)$ that $\rho_*(a,c)\le 0$, that is $\varrho(c)\le R(a)$. It was proved in [3, Lemma 2.3] that the function

$$x\mapsto\frac{\mathcal{K}_a(\sqrt{x})}{R(a)-\log(1-x)}$$

is strictly decreasing on $(0,1)$ for $a\in(0,1/2]$. According to this with Lemma 2.6, we conclude that

$$F(x)=\frac{\mathcal{K}_a(\sqrt{x})}{R(a)-\log(1-x)}\cdot\frac{R(a)-\log(1-x)}{\log(1+c/\sqrt{1-x})}$$

is strictly decreasing on $(0,1)$.

Case 2: $1<c\le c_1(a)$. From Lemma 2.5 and (3.2) we clearly see that $C_{n+1}<C_n$ for $n\ge 1$ and $1<c\le c_2(a)$. By (3.1), it is easy to verify that $C_1\le C_0$. In other words, the sequence $C_n$ is decreasing for $n\in\mathbb{N}_0$. This in conjunction with Lemma 2.1(1) shows that $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,1)$ and so is $H_{f_1,f_2}(x)$ by (2.2). Moreover,

$1 < c \leq c_2(a)$ and (3.4) together with the property of $\rho^*(a,c)$ lead to the conclusion that $\rho^*(a,c) < 0$. Combining this with (3.6) and the monotonicity of $H_{f_1,f_2}(x)$ yields $H_{f_1,f_2}(x) < 0$ for $x \in (0,1)$. Due to $f_2'(x) > 0$ and (2.1), we conclude that $F(x)$ is strictly decreasing on $(0,1)$.

□

***Remark 3.1*** We claim that $\max_{a\in(0,1/2]}\{c_*(a), c_1(a)\} < \max_{a\in(0,1/2]}\{c^*(a), e^{R(a)/2}\}$.

From (3.3), (3.4) and Lemma 2.5 we clearly see that $c_*(a) < c^*(a)$ and $c_1(a) < c_2(a) < c^*(a)$ for $a \in (0,1/2]$. On the other hand, $\rho_*(e^{R(a)/2}; a) = \varrho(e^{R(a)/2}) - R(a) > 0$ follows from

$$\lim_{x\to\infty}\left[\varrho(e^{x/2}) - x\right] = 0 \quad \text{and} \quad \frac{d}{dx}\left[\varrho(e^{x/2}) - x\right] = -e^{-x/2}\log(1+e^{x/2}) < 0.$$

This gives $c_*(a) < e^{R(a)/2}$. From [50, Lemma 2.1] we obtain $1/[a(1-a)] - R(a) < 4 - 4\log 2$, which yields

$$c_2(a) < R(a) + 3 - 4\log 2 < e^{R(a)/2}, \tag{3.8}$$

since $x \mapsto e^{x/2} - x - 3 + 4\log 2$ is strictly increasing on $[4\log 2, \infty)$.

In Lemma 2.5, it only remains to discuss the sign of $\mathcal{Q}_n(a,c)$ for $c_1(a) < c < c_2(a)$. However, in this case, it can be easily seen that $\mathcal{Q}_1(a,c) > 0$ in the proof of Lemma 2.5 and so $C_1 < C_2$ by (3.2), while $C_0 > C_1$ and $C_n > C_{n+1}$ for $n \geq 3$ by Lemma 2.5 and (3.1). This kind of sequence $\{C_n\}_0^\infty$ cannot be handled. So it is not easy to find the sufficient and necessary condition such that $F(x)$ is decreasing on $(0,1)$.

***Remark 3.2*** It is worth pointing out that there is no strict comparison between $c_*(a)$ and $c_1(a)$, and also between $c^*(a)$ and $e^{R(a)/2}$ for $a \in (0,1/2]$.

By the properties of $\rho_*(a,c)$ and $\rho^*(a,c)$, it follows from

$$\lim_{a\to 0^+}\rho_*(a, c_1(a)) = -\infty, \quad \lim_{a\to 1/2}\rho_*(a, c_1(a)) = \frac{4\log 2}{3},$$
$$\lim_{a\to 0^+}\rho^*(a, e^{R(a)/2}) = -1, \quad \lim_{a\to 1/2}\rho^*(a, e^{R(a)/2}) = -4 + \frac{5}{2}\log 5 \approx 0.02359$$

that $c_1(a) < c_*(a)$, $e^{R(a)/2} < c^*(a)$ near $a = 0$ and $c_1(a) > c_*(a)$, $e^{R(a)/2} > c^*(a)$ near $a = 1/2$. More precisely, numerical experiments show that there are two numbers $a_* \approx 0.26372$ and $a^* \approx 0.43722$ such that

- $c_1(a) < c_*(a)$ for $a \in (0, a_*)$ and $c_1(a) > c_*(a)$ for $a \in (a_*, 1/2]$;
- $e^{R(a)/2} < c^*(a)$ for $a \in (0, a^*)$ and $e^{R(a)/2} > c^*(a)$ for $a \in (a^*, 1/2]$.

Following from Ramanujan's asymptotic formula, we can take $c = e^{R(a)/2}$. The following corollary can be derived from Theorem 1.3.

**Corollary 3.1** *The function*

$$r \mapsto \frac{\mathcal{K}_a(r)}{\log\left[1 + e^{R(a)/2}/r'\right]}$$

*is strictly increasing from* $(0,1)$ *onto* $\left(\pi/[2\log(1+e^{R(a)/2})], \sin(a\pi)\right)$ *if and only if* $a^* \approx 0.43722 \leq a \leq 1/2$, *and neither increasing nor decreasing on* $(0,1)$ *if* $a \in (0, a^*)$.

***Proof*** In Remark 3.2, it has been shown that $e^{R(a)/2} \geq c^*(a)$ if and only if $a \in [a^*, 1/2]$.

If $a \in (0, a^*)$, then $e^{R(a)/2} < c^*(a)$, that is $\rho^*(a, e^{R(a)/2}) < 0$. This together with (3.6) and (3.7) gives $H_{f_1,f_2}(0^+) < 0$ and $H_{f_1,f_2}(1^-) = 0$. From (3.8) we clearly see that $e^{R(a)/2} > c_2(a)$. According to this with the proof of Theorem 1.2(1), we know that there exists a number $x_* \in (0, 1)$ such that $H_{f_1,f_2}(x) < 0$ for $x \in (0, x_*)$ and $H_{f_1,f_2}(x) > 0$ for $x \in (x_*, 1)$. So the function is neither increasing nor decreasing on $(0, 1)$. □

Recall that the generalized Grötzsch ring function $\mu_a(r)$ (see [5, (1.3)]) can be defined in the theory of generalized Ramanujan modular equation by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{\mathcal{K}_a(r')}{\mathcal{K}_a(r)}.$$

Due to the monotonicity of $F(r')/F(r)$, the following corollary can be derived immediately from Theorem 1.2.

**Corollary 3.2** *For each* $a \in (0, 1/2]$,

(1) *if* $c \geq \max_{a\in(0,1/2]}\{c^*(a), e^{R(a)/2}\}$, *then the inequality*

$$\frac{\pi^2}{4\sin^2(\pi a)\log(1+c)} \frac{\log(1+c/r)}{\log(1+c/r')} < \mu_a(r) < \log(1+c)\frac{\log(1+c/r)}{\log(1+c/r')}$$

*holds for* $r \in (0, 1)$*;*

(2) *if* $0 < c \leq \max_{a\in(0,1/2]}\{c_*(a), c_1(a)\}$, *then the inequality*

$$\log(1+c)\frac{\log(1+c/r)}{\log(1+c/r')} < \mu_a(r) < \frac{\pi^2}{4\sin^2(\pi a)\log(1+c)} \frac{\log(1+c/r)}{\log(1+c/r')}$$

*holds for* $r \in (0, 1)$.

***Proofs of Theorems 1.3*** Logarithmical differentiating $G(x)$ gives

$$\frac{G'(x)}{G(x)} = -\frac{\lambda}{1-x} + \frac{(1-a)[\mathcal{E}_a - (1-x)\mathcal{K}_a]}{x(1-x)\mathcal{K}_a} = \Phi(\sqrt{x}),$$

where $\Phi$ is defined by Lemma 2.9.

It follow from Lemma 2.6 that $\frac{F'(x)}{F(x)}$ is strictly decreasing if and only if $\lambda \geq \frac{a(1-a)(a^2-a+2)}{2}$ and strictly increasing if and only if $\lambda \leq 0$. Consequently, $F(x)$ is logarithmically concave on $(0, 1)$ if and only if $\lambda \geq \frac{a(1-a)(a^2-a+2)}{2}$ and logarithmically convex on $(0, 1)$ if and only if $\lambda \leq 0$. This completes the proof. □

***Proofs of Theorems 1.4*** By differentiation, we obtain

$$\begin{aligned}
F'(x) &= \frac{(1-a)\left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right]}{x(1-x)\log(1+c/\sqrt{1-x})} - \frac{c\mathcal{K}_a(\sqrt{x})}{2(1-x)(c+\sqrt{1-x})\left[\log(1+c/\sqrt{1-x})\right]^2} \\
&= \frac{2(1-a)\left[\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})\right]\log(1+c/\sqrt{1-x}) - cx\mathcal{K}_a(\sqrt{x})/(c+\sqrt{1-x})}{2x(1-x)[\log(1+c/\sqrt{1-x})]^2} \\
&= (1-a)\frac{\mathcal{E}_a(\sqrt{x}) - (1-x)\mathcal{K}_a(\sqrt{x})}{x} \cdot \frac{1}{\sqrt{1-x}\left(c+\sqrt{1-x}\right)\left[\log(1+c/\sqrt{1-x})\right]^2} \cdot \zeta(\sqrt{x}),
\end{aligned}$$

where $\zeta(r)$ is defined as in Lemma 2.11.

It is not difficult to verify that the function $x \mapsto x(c+x)[\log(1+c/x)]^2$ is strictly increasing and positive on $(0, 1)$. Combining this with Lemmas 2.2(2) and 2.11, we conclude that $F'(x)$ is the product of three positive and increasing functions on $(0, 1)$ if $c \geq \max_{a\in(0,1/2]} \{c^*(a), e^{R(a)/2}\}$. □

Note that $\mathcal{K}_a(\sqrt{2}/2)$ for $a \in (0, 1/2]$ can be expressed as

$$\mathcal{K}_a(\sqrt{2}/2) = \frac{\sin(\pi a)\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})}{4\sqrt{\pi}},$$

which can be found in the literature [5, 4.4 Particular values].

Due to the log-concavity of $G(x)$ and convexity of $F(x)$, we obtain

$$G(x)G(1-x) \leq \sqrt{G(1/2)} = \sqrt{\frac{\sin(\pi a)\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})}{2^{\lambda+2}\sqrt{\pi}}} := \sigma(a, \lambda), \tag{3.9}$$

$$\frac{F(x)+F(1-x)}{2} \geq F(1/2) = \frac{\sin(\pi a)\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})}{4\sqrt{\pi}\log(1+\sqrt{2}c)} := \tau(a, c). \tag{3.10}$$

By substituting $x = r^2$ in (3.9) and (3.10), Theorems 1.3 and 1.4 give rise to the following corollaries.

**Corollary 3.3** *For $\lambda \geq [a(1-a)(a^2-a+2)]/2$, the inequality*

$$(rr')^{2\lambda}\mathcal{K}_a(r)\mathcal{K}_a(r') \leq \sigma(a, \lambda)$$

*holds for $r \in (0, 1)$ with the best constant $\sigma(a, \lambda)$ given in (3.9).*

**Corollary 3.4** *For $c \geq \max_{a\in(0,1/2]} \{c^*(a), e^{R(a)/2}\}$, the inequality*

$$\frac{\mathcal{K}_a(r)}{\log(1+c/r')} + \frac{\mathcal{K}_a(r')}{\log(1+c/r)} \geq 2\tau(a, c)$$

*holds for $r \in (0, 1)$ with the best constant $\tau(a, c)$ given in (3.10), where $c^*(a)$ is defined as in Theorem 1.2.*

Applying L'Hôpital Monotone Rule, it follows easily from Theorem 1.4 that

$$\frac{F(x)-F(0)}{x} \quad \text{and} \quad \frac{F(1)-F(x)}{1-x} \tag{3.11}$$

are increasing on $(0, 1)$ for $c \geq \max_{a\in(0,1/2]} \{c^*(a), e^{R(a)/2}\}$, which gives the following corollary by putting $x = r^2$ in (3.11).

**Corollary 3.5** *For $c \geq \max_{a\in(0,1/2]} \{c^*(a), e^{R(a)/2}\}$, both of the functions*

$$r \mapsto \frac{1}{r^2}\left[\frac{\mathcal{K}_a(r)}{\log(1+c/r')} - \frac{\pi}{2\log(1+c)}\right] \quad \textit{and} \quad r \mapsto \frac{1}{r'^2}\left[\sin(\pi a) - \frac{\mathcal{K}_a(r)}{\log(1+c/r')}\right]$$

*are strictly increasing on $(0, 1)$. Consequently, the double inequality*

$$\frac{\pi}{2\log(1+c)} + \frac{\pi\left[a(1-a)\varrho(c)-1\right]}{4(1+c)\left[\log(1+c)\right]^2}r^2 < \frac{\mathcal{K}_a(r)}{\log(1+c/r')} < \sin(\pi a)r^2 + \frac{\pi}{2\log(1+c)}r'^2$$

*holds for $r \in (0, 1)$, where $\varrho(c)$ is defined as in Lemma 2.6.*

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